# Discrete-Time Bases and Filter Banks

Advances Signal Processing Seminar

Stefan Mendel & Franz Zotter





## Outline

- Introduction
  - Orthonormality
  - Biorthogonality
- Orthonormal expansions and filter banks
  - Haar expansion
  - Sinc expansion
- Analysis of filter banks
  - Time domain
  - Modulation domain
  - Polyphase domain
  - Relations between time, modulation, and polyphase domain
- Results on filter banks
  - Biorthogonal Relations
  - Orthogonal Relations

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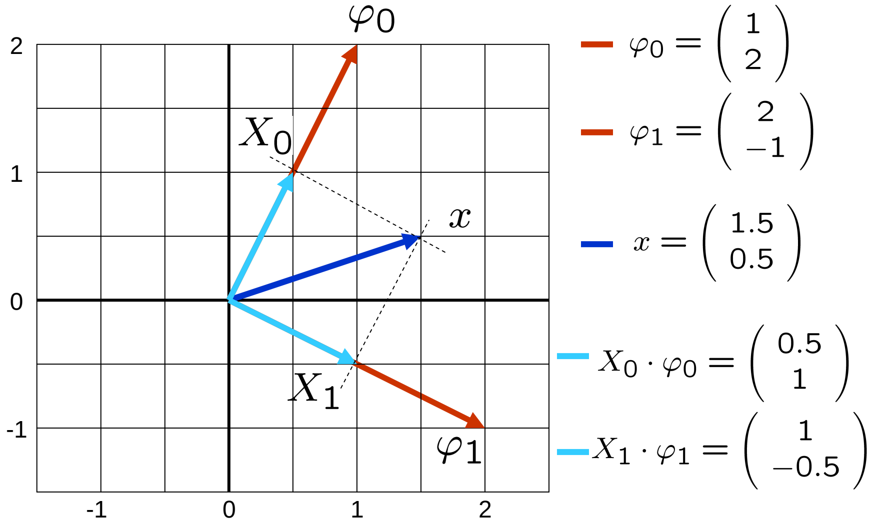
# Orthonormal Expansion

- Sequence $x[n]$ is square-summable $x[n] \in l_2(\mathcal{Z})$
- Expansion $x[n] = \sum_{k \in \mathcal{Z}} \langle \varphi_k[l], x[l] \rangle \varphi_k[n] = \sum_{k \in \mathcal{Z}} X[k] \varphi_k[n]$
- Transform $X[k] = \langle \varphi_k[l], x[l] \rangle = \sum_{l} \varphi_k^*[l] x[l]$
- Orthonormality $\langle \varphi_k[n], \varphi_l[n] \rangle = \delta[k-l]$
- Conservation of energy $\|x\|^2 = \|X\|^2$

# Orthogonal: Example







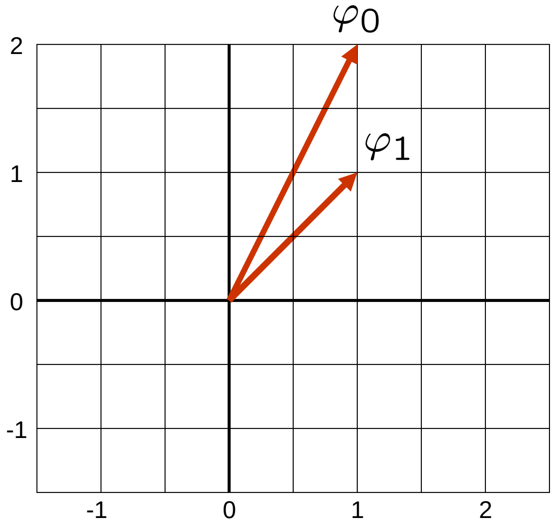

# Biorthogonal Expansion

- Expansion
$$x[n] = \sum_{k\in\mathcal{Z}} \langle \varphi_k[l], x[l] \rangle \tilde{\varphi}_k[n] = \sum_{k\in\mathcal{Z}} \tilde{X}[k]\tilde{\varphi}_k[n]$$
$$= \sum_{k\in\mathcal{Z}} \langle \tilde{\varphi}_k[l], x[l] \rangle \varphi_k[n] = \sum_{k\in\mathcal{Z}} X[k]\varphi_k[n]$$

- Transform $\tilde{X}[k] = \langle \varphi_k[l], x[l] \rangle$ and $X[k] = \langle \tilde{\varphi}_k[l], x[l] \rangle$

- Conservation of energy $\|x\|^2 = \langle X[k], \tilde{X}[k] \rangle$

# Biorthogonal: Example



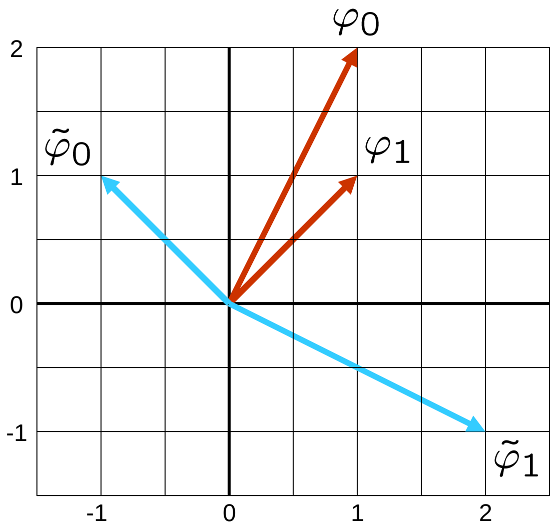

$\varphi_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$





# Biorthogonal: Example



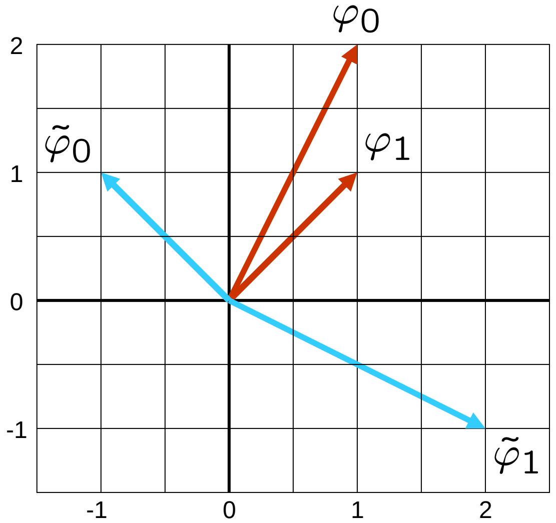

$\varphi_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\tilde{\varphi}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\tilde{\varphi}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

# Biorthogonal: Example




$$< \varphi_k, \tilde{\varphi}_l > = \delta(k-l)$$

$$< \varphi_0, \tilde{\varphi}_0 > = 1$$
$$< \varphi_0, \tilde{\varphi}_1 > = 0$$
$$< \varphi_1, \tilde{\varphi}_0 > = 0$$
$$< \varphi_1, \tilde{\varphi}_1 > = 1$$



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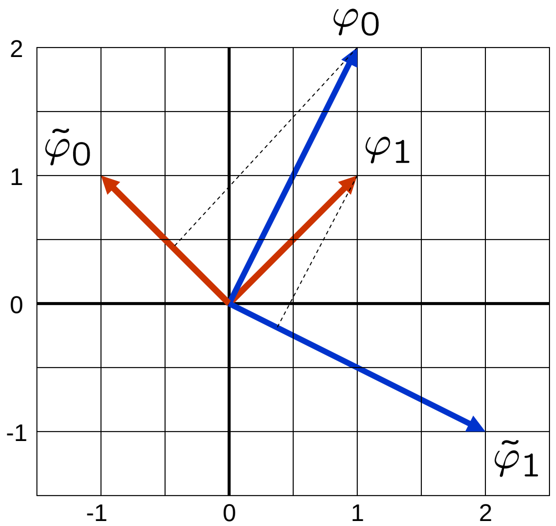
# Biorthogonal: Example

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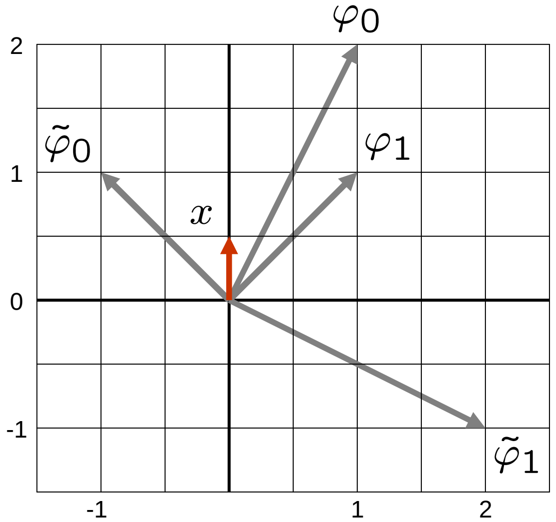
$$< \varphi_1, \tilde{\varphi}_0 > = 0$$
$$< \varphi_1, \tilde{\varphi}_1 > = 1$$

$$\varphi_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\tilde{\varphi}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \tilde{\varphi}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

# Biorthogonal: Example Reconstruction




$$x = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$

$$x[n] = \sum_{k \in \mathcal{Z}} X[k] \varphi_k[n]$$

$$X[k] = \langle \tilde{\varphi}_k[l], x[l] \rangle$$

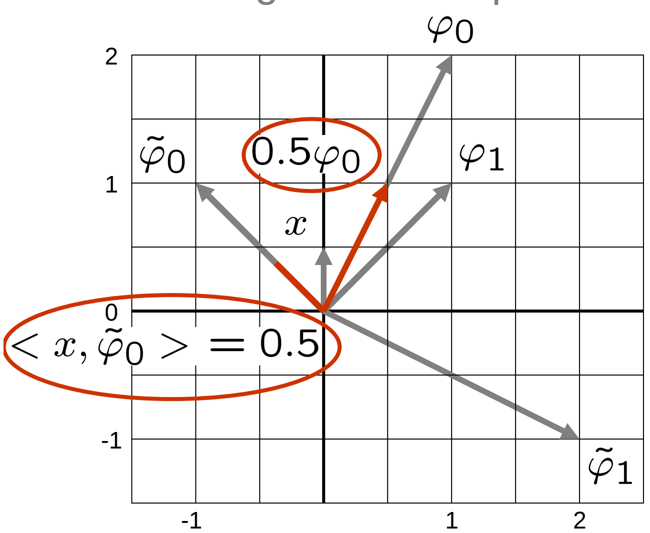
$$x[n] = \sum_{k \in \mathcal{Z}} \tilde{X}[k] \tilde{\varphi}_k[n]$$

$$\tilde{X}[k] = \langle \varphi_k[l], x[l] \rangle$$





# Biorthogonal: Example Reconstruction

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$$x[n] = \sum_{k \in \mathcal{Z}} X[k] \varphi_k[n]$$

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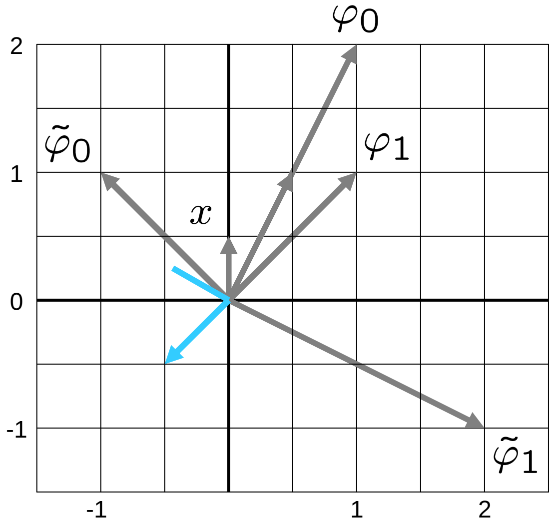
$$X[0] = 0.5$$

$$X[1] = -0.5$$

# Biorthogonal: Example Reconstruction

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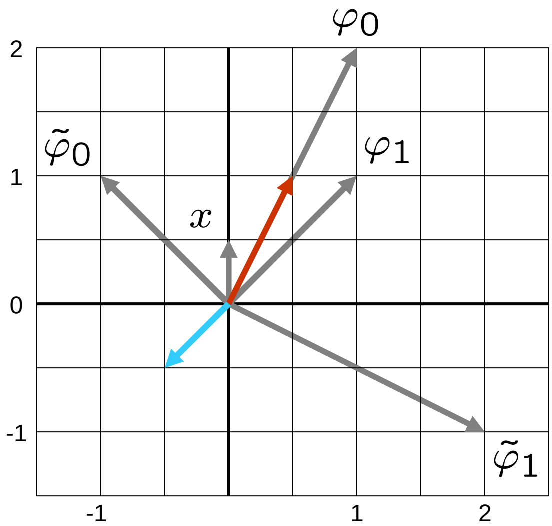
# Biorthogonal: Example Reconstruction

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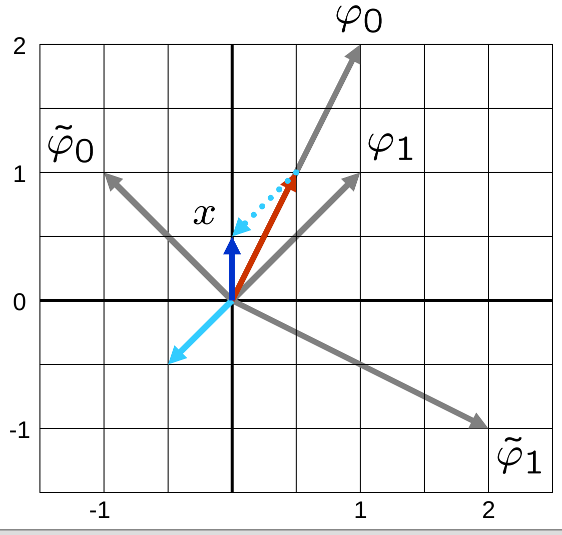


$$x = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$

$$x[n] = \sum_{k \in \mathcal{Z}} X[k] \varphi_k[n]$$

$$X[k] = \langle \tilde{\varphi}_k[l], x[l] \rangle$$

$$X[0] = 0.5$$

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# Haar Expansion

- Basis functions

$$\varphi_{2k}[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = 2k, 2k+1, \\ 0 & \text{otherwise}, \end{cases} \quad \varphi_{2k+1}[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = 2k, \\ -\frac{1}{\sqrt{2}} & n = 2k+1, \\ 0 & \text{otherwise}. \end{cases}$$

- Time-varying periodic

$$\varphi_{2k}[n] = \varphi_0[n-2k], \quad \varphi_{2k+1}[n] = \varphi_1[n-2k]$$

- Transform

$$X[2k] = \langle \varphi_{2k}, x \rangle = \frac{1}{\sqrt{2}}\Big(x[2k] + x[2k+1]\Big)$$
$$X[2k+1] = \langle \varphi_{2k+1}, x \rangle = \frac{1}{\sqrt{2}}\Big(x[2k] - x[2k+1]\Big)$$





# Haar Expansion & Filterbanks

- Filter

$$h_0[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = -1, 0, \\ 0 & \text{otherwise}, \end{cases} \quad h_1[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = 0, \\ -\frac{1}{\sqrt{2}} & n = -1, \\ 0 & \text{otherwise}. \end{cases}$$

$$h_0[n] * x[n]\Big|_{n=2k} = \sum_{l \in \mathcal{Z}} h_0[2k-l]x[l] = \frac{1}{\sqrt{2}}x[2k] + \frac{1}{\sqrt{2}}x[2k+1] = X[2k]$$

$$h_1[n] * x[n]\Big|_{n=2k} = \sum_{l \in \mathcal{Z}} h_1[2k-l]x[l] = \frac{1}{\sqrt{2}}x[2k] - \frac{1}{\sqrt{2}}x[2k+1] = X[2k+1]$$

Filters $h_0[n]$ and $h_1[n]$ followed by downsampling by 2 implement $\varphi_0$ and $\varphi_1$

$$\boxed{h_0[n] = \varphi_0[-n], h_1[n] = \varphi_1[-n]}$$

## Time-Domain Analysis

$$
\begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[1] \\ y_1[1] \\ \vdots \end{pmatrix}
=
\begin{pmatrix} \vdots \\ X[0] \\ X[1] \\ X[2] \\ X[3] \\ \vdots \end{pmatrix}
=
\begin{pmatrix} \ddots & & & \\ & \overbrace{\begin{matrix} h_0[0] h_0[-1] \\ h_1[0] h_1[-1] \end{matrix}}^{\varphi_0[n]}_{\underbrace{\qquad\qquad}_{\varphi_1[n]}} & & \\ & & \overbrace{\begin{matrix} h_0[0] h_0[-1] \\ h_1[0] h_1[-1] \end{matrix}}^{\varphi_0[n]}_{\underbrace{\qquad\qquad}_{\varphi_1[n]}} & \\ & & & \ddots \end{pmatrix}
\begin{pmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \end{pmatrix}
$$





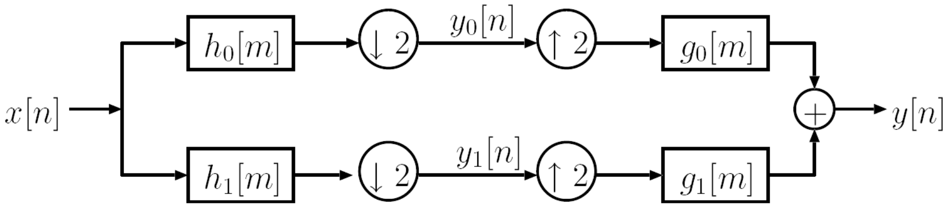
## Reconstruction

- Filter $g_0[n] = \varphi_0[n], g_1[n] = \varphi_1[n]$
- Periodic $\varphi_{2k}[n] = g_0[n-2k], \varphi_{2k+1}[n] = g_1[n-2k]$

$$
\begin{aligned}
x[n] &= \sum_{k \in \mathcal{Z}} X[k] \varphi_k[n] \\
&= \sum_{k \in \mathcal{Z}} X[2k] \varphi_{2k}[n] + \sum_{k \in \mathcal{Z}} X[2k+1] \varphi_{2k+1}[n] \\
&= \sum_{k \in \mathcal{Z}} y_0[k] g_0[n-2k] + \sum_{k \in \mathcal{Z}} y_1[k] g_1[n-2k]
\end{aligned}
$$

Upsampling by 2 followed by convolution with $g_i$

# Filterbank



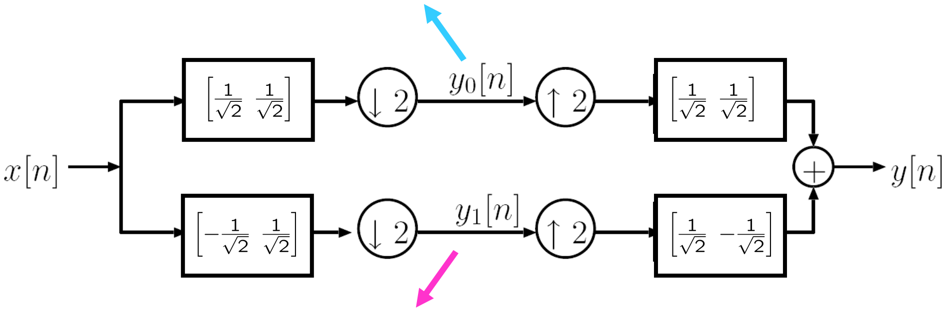

- Synthesis Filter $g_i[n] = \varphi_i[n]$
- Analysis Filter $h_i[n] = \varphi_i[-n]$





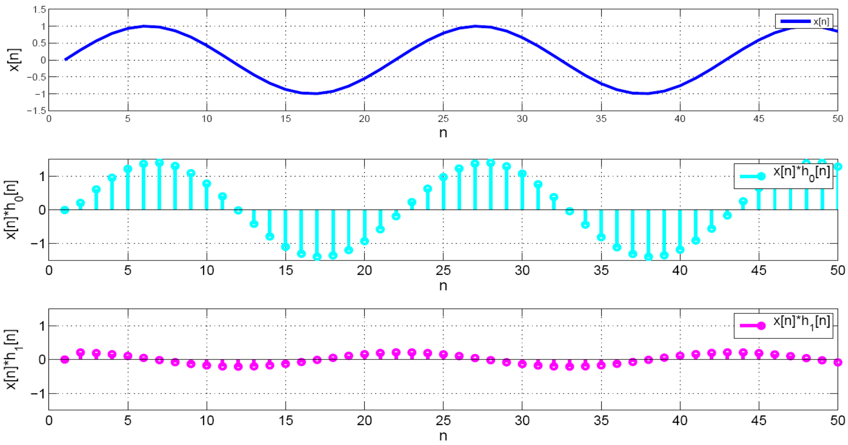
# Filterbank

$$y_0[k] = X[2k] = \frac{1}{\sqrt{2}}x[2k] + \frac{1}{\sqrt{2}}x[2k+1]$$




$$y_1[k] = X[2k+1] = \frac{1}{\sqrt{2}}x[2k] - \frac{1}{\sqrt{2}}x[2k+1]$$

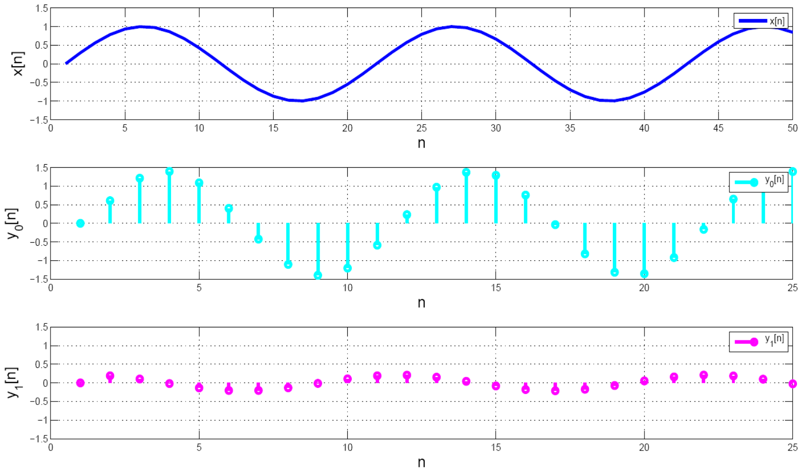
# Expansion Example – Analysis Filter





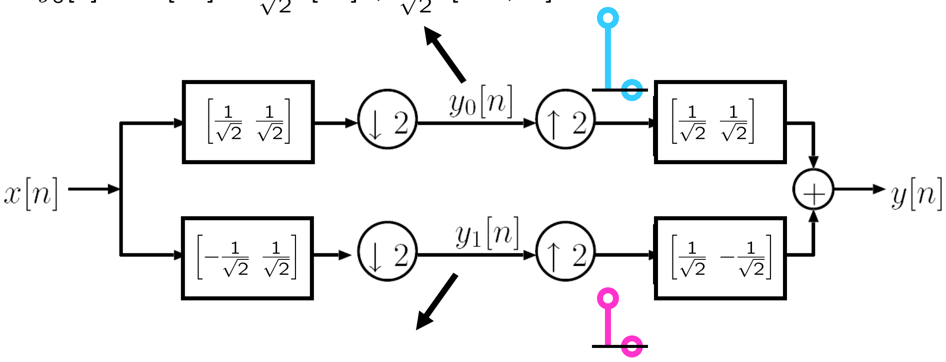



# Haar Example - Downsampling



# Filterbank

$$y_0[k] = X[2k] = \frac{1}{\sqrt{2}}x[2k] + \frac{1}{\sqrt{2}}x[2k+1]$$

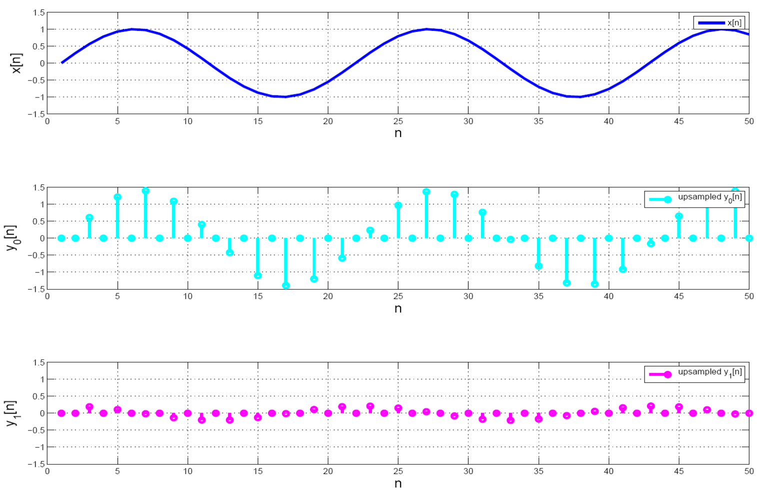



$$y_1[k] = X[2k+1] = \frac{1}{\sqrt{2}}x[2k] - \frac{1}{\sqrt{2}}x[2k+1]$$



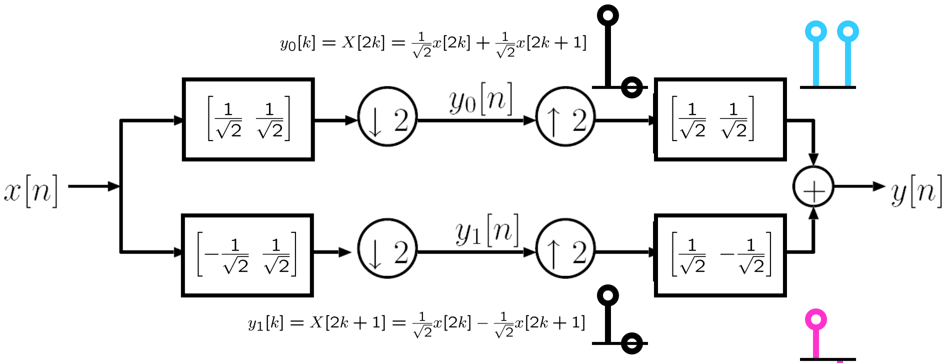


# Haar Example - Upsampling



# Filterbank

$$x_1[2k] = x_1[2k+1] = \frac{1}{2}x[2k] + \frac{1}{2}x[2k+1]$$




$$x_1[2k] = \frac{1}{2}x[2k] - \frac{1}{2}x[2k+1]$$

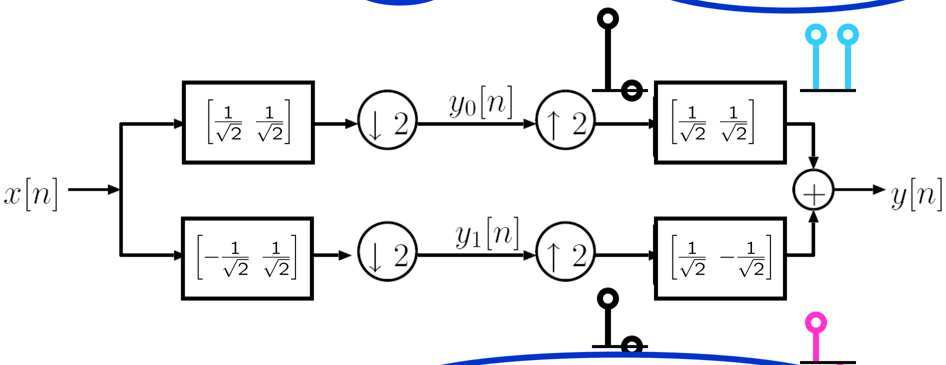
$$x_1[2k+1] = -\frac{1}{2}x[2k] + \frac{1}{2}x[2k+1]$$





# Filterbank

$$x_1[2k] = x_1[2k+1] = \frac{1}{2}x[2k] + \frac{1}{2}x[2k+1]$$

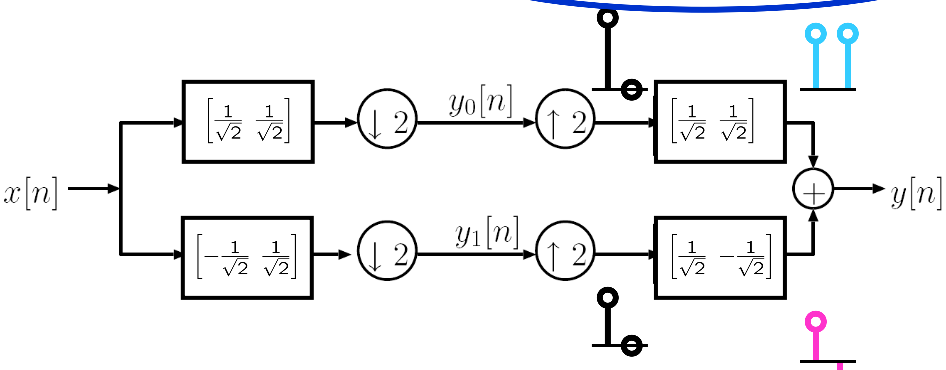



$$x_1[2k] = \frac{1}{2}x[2k] - \frac{1}{2}x[2k+1]$$

$$x_1[2k+1] = -\frac{1}{2}x[2k] + \frac{1}{2}x[2k+1]$$

# Filterbank

$$x_1[2k] = x_1[2k+1] = \tfrac{1}{2}x[2k] + \tfrac{1}{2}x[2k+1]$$



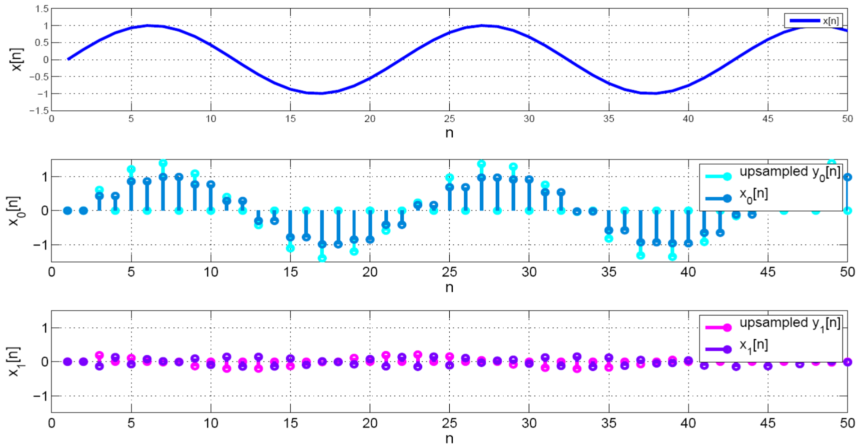

$$x_1[2k] = \tfrac{1}{2}x[2k] - \tfrac{1}{2}x[2k+1]$$

$$x_1[2k+1] = -\tfrac{1}{2}x[2k] + \tfrac{1}{2}x[2k+1]$$

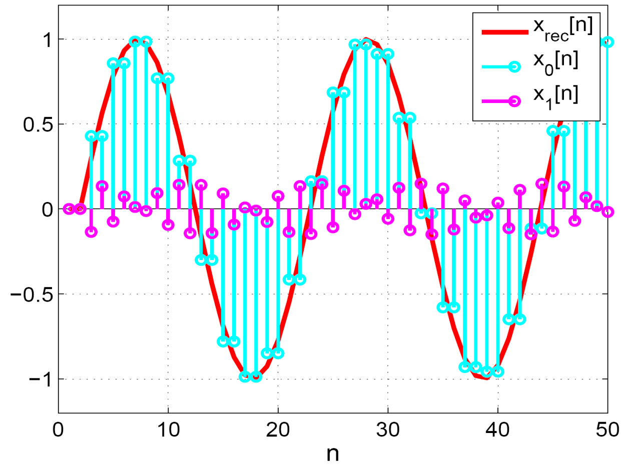




# Haar Example – Synthesis Filter



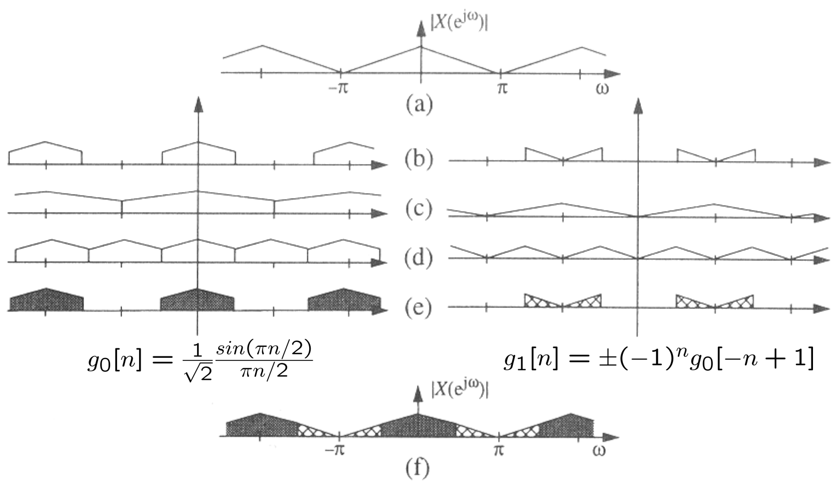
# Haar Example - Reconstruction








# Sinc Expansion




$$g_0[n] = \frac{1}{\sqrt{2}} \frac{sin(\pi n/2)}{\pi n/2}$$

$$g_1[n] = \pm(-1)^n g_0[-n+1]$$

## Orthogonal Expansions - Summary

- Synthesis filter $g_i[n] = \varphi_i[n]$
- Analysis filter $h_i[n] = g_i[-n] = \varphi_i[-n]$
- Expansions are periodically time- varying
- Haar expansion
  - Good time resolution
- Sinc expansion
  - Good frequency resolution





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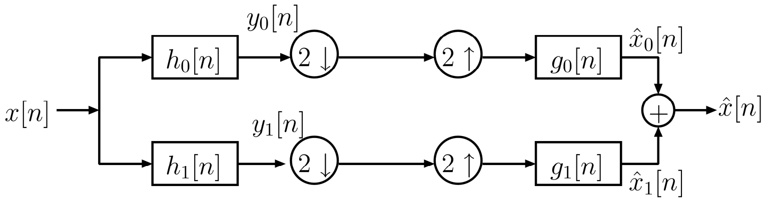
# Analysis of Filter Banks: Time Domain




- Analysis: $y_k[n] = x[n] \star h_k[n] \quad = \; < x[n], h_k[-n] >$

  $\Rightarrow h_k[-n] = \tilde{\varphi}_k[n]$, i.e. non-causal filter

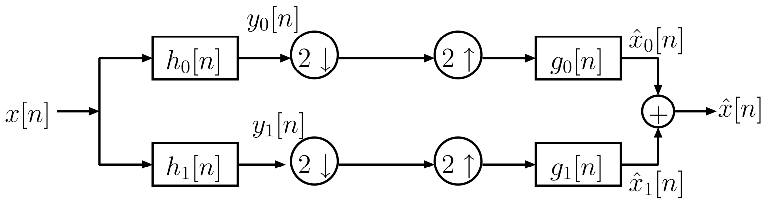
- Synthesis: (from decimated $y_k$) $\hat{x}[bN+n] = \sum_{k=0}^{N} \sum_{m=b-n/N}^{\frac{L-1-n}{N}-b} y_k[lN]\cdot g_k[mN-bN+n]$

  $\Rightarrow g_k[n] = \varphi_k[n]$





# Analysis of Filter Banks: Time Domain




- Synthesis/analysis: decimated, interlaced channels:

Analysis: $\tilde{\mathbf{X}} = \mathbf{T_a} \cdot \mathbf{x}$

$$\begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[2] \\ y_1[2] \\ \vdots \end{pmatrix} = \mathbf{T}_a \cdot \begin{pmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \end{pmatrix}$$

Synthesis: $\mathbf{y} = \mathbf{T_s} \cdot \tilde{\mathbf{X}}$

$$\begin{pmatrix} \vdots \\ \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \\ \vdots \end{pmatrix} = \mathbf{T}_s \cdot \begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[2] \\ y_1[2] \\ \vdots \end{pmatrix}$$

# Analysis of Filter Banks: Time Domain

- Decimated, interlaced: Analysis

$$\underbrace{\begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[2] \\ y_1[2] \\ \vdots \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & h_0[0] & h_0[-1] & h_0[-2] & \dots & h_0[1-L] & 0 & 0 & 0 & \dots \\ \dots & 0 & h_1[0] & h_1[-1] & h_1[-2] & \dots & h_1[1-L] & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & h_0[0] & \dots & h_0[3-L] & h_0[2-L] & h_0[1-L] & 0 & \dots \\ \dots & 0 & 0 & 0 & h_1[0] & \dots & h_1[3-L] & h_1[2-L] & h_1[1-L] & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{\mathbf{T_a}} \underbrace{\begin{pmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \end{pmatrix}}_{\mathbf{x}}$$

- Synthesis:

$$\overbrace{\begin{pmatrix} \vdots \\ \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \\ \vdots \end{pmatrix}}^{\hat{\mathbf{x}}} = \overbrace{\begin{pmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & g_0[0] & g_1[0] & 0 & 0 & \dots \\ \dots & g_0[1] & g_1[1] & 0 & 0 & \dots \\ \dots & g_0[2] & g_1[2] & g_0[0] & g_1[0] & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & g_0[L-1] & g_1[L-1] & g_0[L-3] & g_1[L-3] & \dots \\ \dots & 0 & 0 & g_0[L-2] & g_1[L-2] & \dots \\ \dots & 0 & 0 & g_0[L-1] & g_1[L-1] & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}}^{\mathbf{T_s}} \overbrace{\begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[2] \\ y_1[2] \\ \vdots \end{pmatrix}}^{\mathbf{y}}$$





# Analysis of Filter Banks: Time Domain

- Decimated, interlaced: Analysis $\tilde{\varphi}_0[n]$ $\tilde{\varphi}_1[n]$

$$\underbrace{\begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[2] \\ y_1[2] \\ \vdots \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & h_0[0] & h_0[-1] & h_0[-2] & \dots & h_0[1-L] & 0 & 0 & 0 & \dots \\ \dots & 0 & h_1[0] & h_1[-1] & h_1[-2] & \dots & h_1[1-L] & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & h_0[0] & \dots & h_0[3-L] & h_0[2-L] & h_0[1-L] & 0 & \dots \\ \dots & 0 & 0 & 0 & h_1[0] & \dots & h_1[3-L] & h_1[2-L] & h_1[1-L] & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{\mathbf{T_a}} \underbrace{\begin{pmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \end{pmatrix}}_{\mathbf{x}}$$

- Synthesis: $\varphi_0[n]$ $\varphi_1[n]$

$$\overbrace{\begin{pmatrix} \vdots \\ \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \hat{x}[3] \\ \vdots \end{pmatrix}}^{\hat{\mathbf{x}}} = \overbrace{\begin{pmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & g_0[0] & g_1[0] & 0 & 0 & \dots \\ \dots & g_0[1] & g_1[1] & 0 & 0 & \dots \\ \dots & g_0[2] & g_1[2] & g_0[0] & g_1[0] & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & g_0[L-1] & g_1[L-1] & g_0[L-3] & g_1[L-3] & \dots \\ \dots & 0 & 0 & g_0[L-2] & g_1[L-2] & \dots \\ \dots & 0 & 0 & g_0[L-1] & g_1[L-1] & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}}^{\mathbf{T_s}} \overbrace{\begin{pmatrix} \vdots \\ y_0[0] \\ y_1[0] \\ y_0[2] \\ y_1[2] \\ \vdots \end{pmatrix}}^{\mathbf{y}}$$

# Analysis of Filter Banks: Time Domain

- Perfect reconstruction: $\mathbf{T_s T_a = I}$

(Biorthogonality)

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & h_0[0] & h_0[-1] & h_0[-2] & \dots & h_0[1-L] & 0 & 0 & 0 & \dots \\ \dots & 0 & h_1[0] & h_1[-1] & h_1[-2] & \dots & h_1[1-L] & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & h_0[0] & \dots & h_0[3-L] & h_0[2-L] & h_0[1-L] & 0 & \dots \\ \dots & 0 & 0 & 0 & h_1[0] & \dots & h_1[3-L] & h_1[2-L] & h_1[1-L] & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \times$$

$$\begin{pmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & g_0[0] & g_1[0] & 0 & 0 & \dots \\ \dots & g_0[1] & g_1[1] & 0 & 0 & \dots \\ \dots & g_0[2] & g_1[2] & g_0[0] & g_1[0] & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & g_0[L-1] & g_1[L-1] & g_0[L-3] & g_1[L-3] & \dots \\ \dots & 0 & 0 & g_0[L-2] & g_1[L-2] & \dots \\ \dots & 0 & 0 & g_0[L-1] & g_1[L-1] & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} = \mathbf{I}$$

- Orthonormality:
  - Analysis filters are time reversed synthesis filters

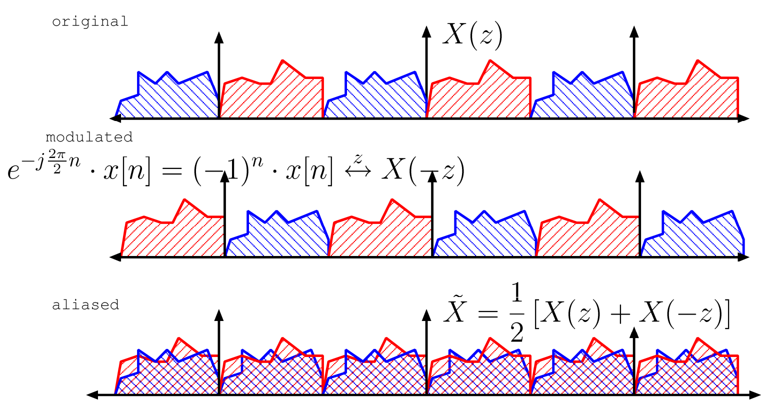
$\mathbf{T_a = T_s^T}$, i.e. $h_k[n] = g_k[-n]$, and $\mathbf{T_s^T T_s = I}$





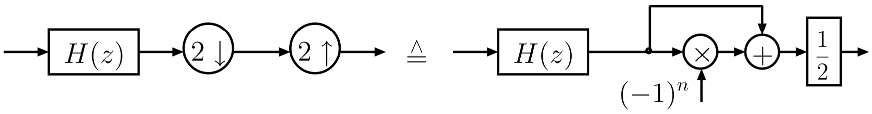
# Analysis of Filter Banks: Modulation Domain

- Aliased spectra by modulation: A decimation by 2 example



# Analysis of Filter Banks: Modulation Domain

- Aliased spectra by modulation: A decimation by 2 example
  - Replacing decimation and upsampling by modulation

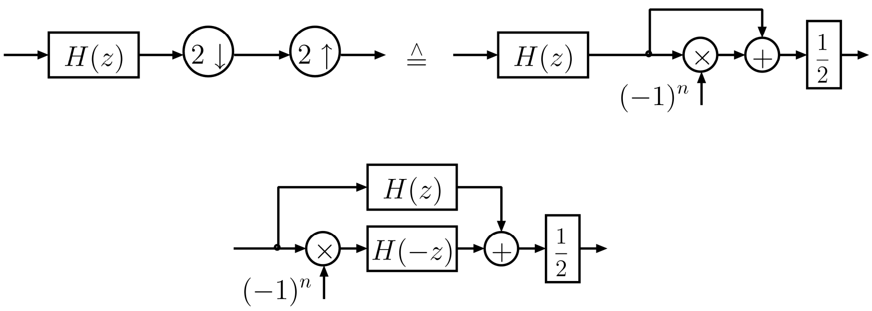






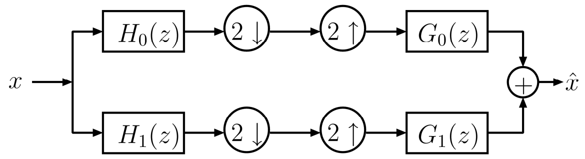

# Analysis of Filter Banks: Modulation Domain

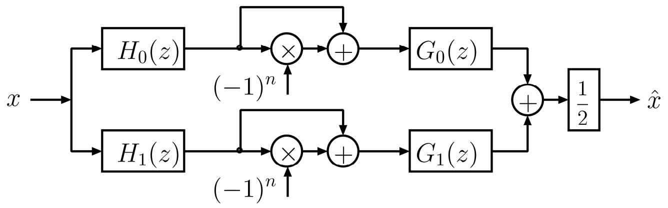
- Aliased spectra by modulation: A decimation by 2 example
  - Replacing decimation and upsampling by modulation
  - Employing modulated versions of the filter



# Analysis of Filter Banks: Modulation Domain

- A 2-channel example:
  - Replacing decimation+upsamling by modulation



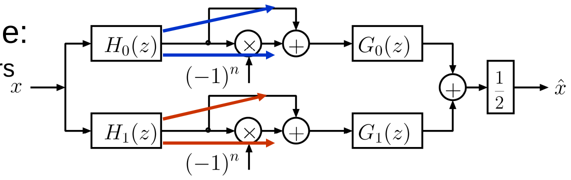



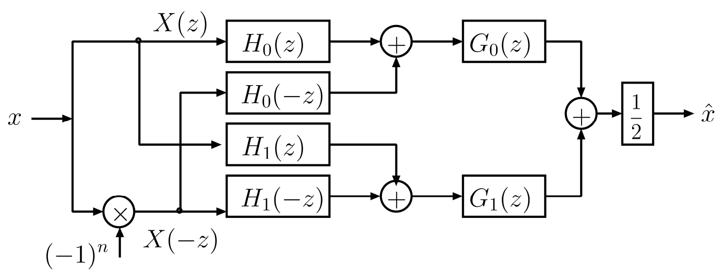





# Analysis of Filter Banks: Modulation Domain

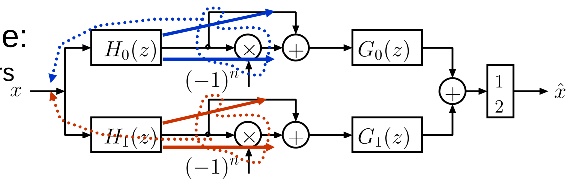
- A 2-channel example:
  - Pulling modulated filters into modulation path

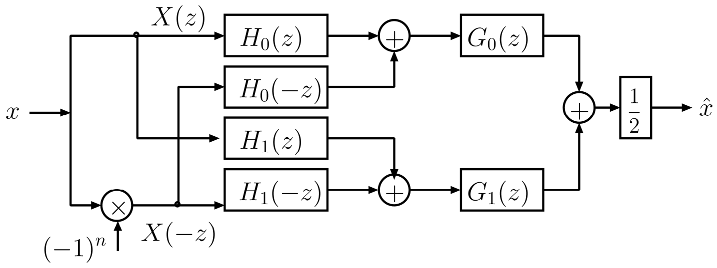


# Analysis of Filter Banks: Modulation Domain

- A 2-channel example:
  - Pulling modulated filters into modulation path










# Analysis of Filter Banks: Modulation Domain

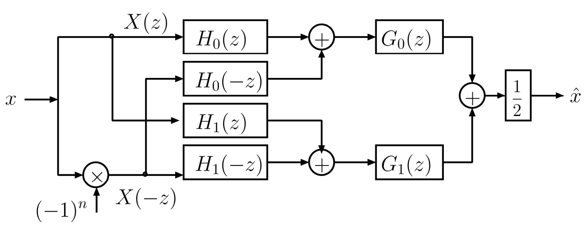
- A 2-channel example:
  - We finally get the system as matrix of modulated filters

Analysis:

$$\mathbf{Y}(z) = \frac{1}{2} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix}$$

Synthesis:

$$\widehat{X}(z) = \begin{pmatrix} G_0(z) & G_1(z) \end{pmatrix} \mathbf{Y}(z)$$



# Analysis of Filter Banks: Modulation Domain

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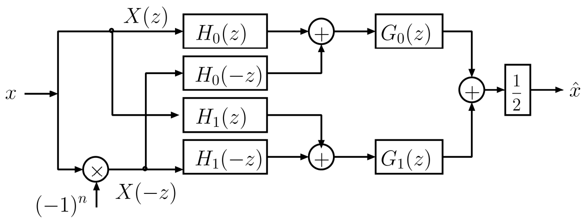






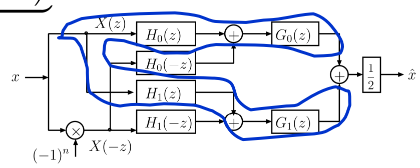

# Analysis of Filter Banks: Modulation Domain

- Perfect reconstruction:
(Biorthogonality)

$$\frac{1}{2}\begin{pmatrix} G_0(z) & G_1(z) \end{pmatrix} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix} \stackrel{!}{=} X(z)$$

$$\Rightarrow \frac{1}{2}\begin{pmatrix} G_0(z) & G_1(z) \end{pmatrix} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

1:1 transfer function




- Orthonormality:

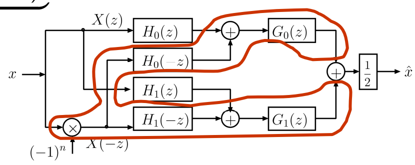
# Analysis of Filter Banks: Modulation Domain

- Perfect reconstruction:
  (Biorthogonality)

$$\frac{1}{2}\begin{pmatrix} G_0(z) & G_1(z) \end{pmatrix} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix} \stackrel{!}{=} X(z)$$

$$\Rightarrow \frac{1}{2}\begin{pmatrix} G_0(z) & G_1(z) \end{pmatrix} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

no aliasing




- Orthonormality:



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Analysis of Filter Banks: Modulation Domain

- Perfect reconstruction:
  (Biorthogonality)

$$\underbrace{\frac{1}{2}\begin{pmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{pmatrix}}_{\mathbf{G_m}} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\underbrace{\begin{pmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{pmatrix}}_{\mathbf{G_m}} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} = \mathbf{I}$$

with modulated synthesis filters:
- elegant notation!

$$\frac{1}{2}\mathbf{G_m}(z)\mathbf{H_m}(z) = \mathbf{I}$$

- Orthonormality:

# Analysis of Filter Banks: Modulation Domain

- Perfect reconstruction:
  (Biorthogonality)

$$\frac{1}{2}\underbrace{\begin{pmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{pmatrix}}_{\mathbf{G_m}} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} X(z) \\ X(-z) \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\underbrace{\begin{pmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{pmatrix}}_{\mathbf{G_m}} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{\mathbf{H_m}} = \mathbf{I}$$

with modulated synthesis filters:
- elegant notation!

$$\frac{1}{2}\mathbf{G_m}(z)\mathbf{H_m}(z) = \mathbf{I}$$

- Orthonormality:
  - Analysis filters are time reversed synthesis filters

$$\mathbf{H_m}(z) = \mathbf{G_m^T}(z^{-1}) \qquad \frac{1}{2}\mathbf{G_m}(z)\mathbf{G_m^T}(z^{-1}) = \mathbf{I}$$

$\{\}^{\mathbf{T}}$ is the hermitian transpose

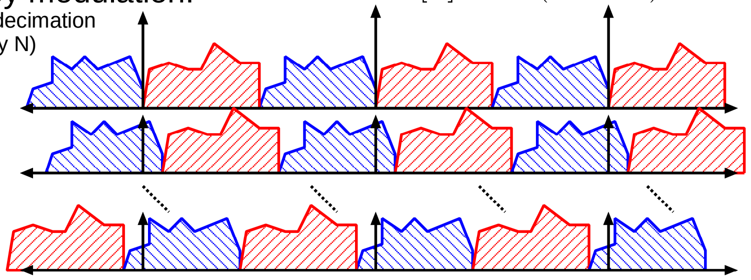




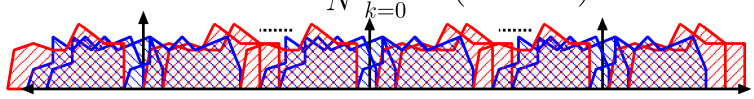
# Analysis of Filter Banks: Modulation Domain

- Aliased spectra by modulation:
  (decimation by N)

$$e^{-j\frac{2\pi}{L}k} \cdot x[n] \stackrel{z}{\leftrightarrow} X(ze^{-j\frac{2\pi}{L}k})$$

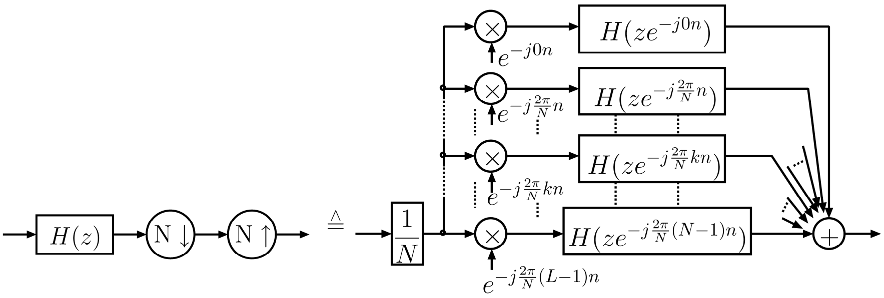


$$\tilde{X}(z) = \frac{1}{N}\sum_{k=0}^{L-1} X\left(ze^{-j\frac{2\pi}{L}kn}\right)$$



# Analysis of Filter Banks: Modulation Domain

- Aliased spectra
  by modulation: single filter, decimation by N
  - Replacing decimation and upsampling by modulation
  - Pulling filters into modulation paths

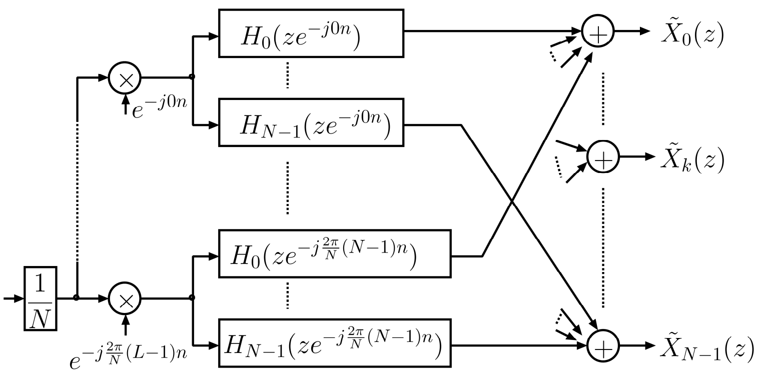







# Analysis of Filter Banks: Modulation Domain

- Aliased spectra
  by modulation: N channel filter bank
  - Modulation domain for N-channel filter banks



# Analysis of Filter Banks: Modulation Domain

- Perfect reconstruction: arbitrary N-channel case

  (Biorthogonality)

with modulated synthesis filters

$$\mathbf{G_m} = \begin{pmatrix} G_0\left(e^{-j\frac{2\pi}{N}0}z\right) & \dots & G_{N-1}\left(e^{-j\frac{2\pi}{N}0}z\right) \\ \vdots & \dots & \vdots \\ G_0\left(e^{-j\frac{2\pi}{N}(N-1)}z\right) & \dots & G_{N-1}\left(e^{-j\frac{2\pi}{N}(N-1)}z\right) \end{pmatrix}$$

$$\frac{1}{N}\mathbf{G_m}(z)\mathbf{H_m}(z) \cdot \begin{pmatrix} X(e^{-j\frac{2\pi}{N}0}z) \\ \vdots \\ X(e^{-j\frac{2\pi}{N}(N-1)}z) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} X(e^{-j\frac{2\pi}{N}0}z) \\ \vdots \\ X(e^{-j\frac{2\pi}{N}(N-1)}z) \end{pmatrix}$$

$$\frac{1}{N}\mathbf{G_m} \cdot \mathbf{H_m} = \mathbf{I}$$

- Orthonormality:



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# Analysis of Filter Banks: Modulation Domain

- Perfect reconstruction: arbitrary N-channel case

  (Biorthogonality)

with modulated synthesis filters

$$\mathbf{G_m} = \begin{pmatrix} G_0\left(e^{-j\frac{2\pi}{N}0}z\right) & \dots & G_{N-1}\left(e^{-j\frac{2\pi}{N}0}z\right) \\ \vdots & \dots & \vdots \\ G_0\left(e^{-j\frac{2\pi}{N}(N-1)}z\right) & \dots & G_{N-1}\left(e^{-j\frac{2\pi}{N}(N-1)}z\right) \end{pmatrix}$$

$$\frac{1}{N}\mathbf{G_m}(z)\mathbf{H_m}(z) \cdot \begin{pmatrix} X(e^{-j\frac{2\pi}{N}0}z) \\ \vdots \\ X(e^{-j\frac{2\pi}{N}(N-1)}z) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} X(e^{-j\frac{2\pi}{N}0}z) \\ \vdots \\ X(e^{-j\frac{2\pi}{N}(N-1)}z) \end{pmatrix}$$

$$\frac{1}{N}\mathbf{G_m} \cdot \mathbf{H_m} = \mathbf{I}$$

- Orthonormality:
  - Analysis filters are time reversed synthesis filters

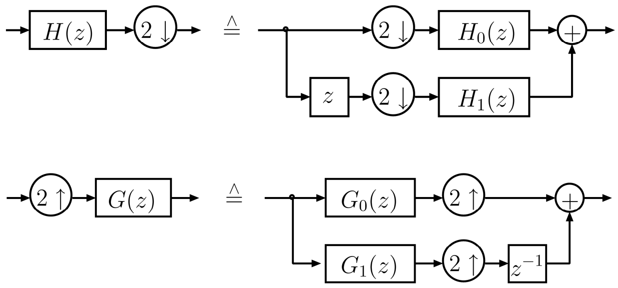
$$\mathbf{H_m}(z) = \mathbf{G_m^T}(z^{-1}) \qquad \frac{1}{N}\mathbf{G_m}(z)\mathbf{G_m^T}(z^{-1}) = \mathbf{I}$$

$\{\}^{\mathbf{T}}$ is the hermitian transpose

# Analysis of Filter Banks: Polyphase Domain

- Polyphase implementation of anti-aliasing and interpolation filters: A decimation by 2 example

(recall Mr. Saleem's talk in 1st session)

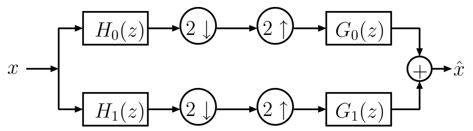


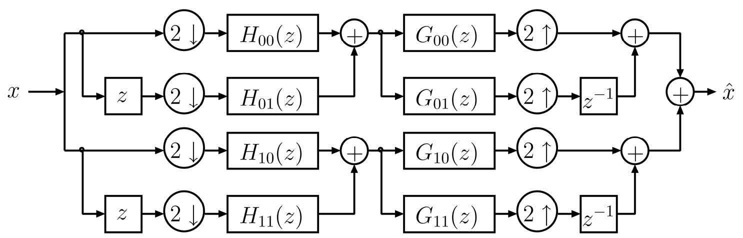





# Analysis of Filter Banks: Polyphase Domain

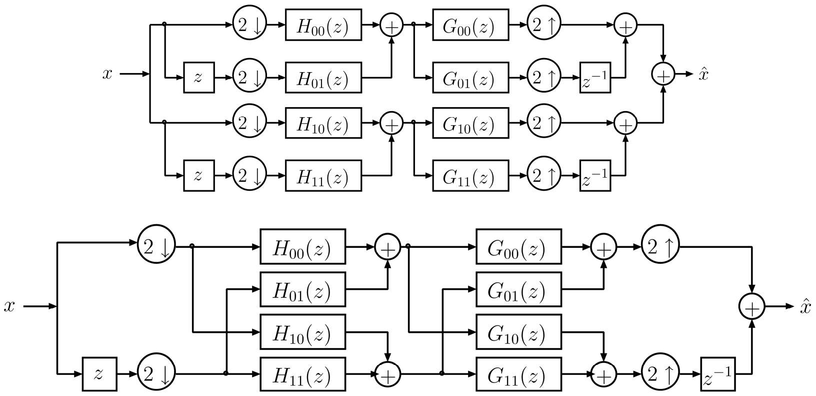
- Decimation and upsampling: 2-channel example



# Analysis of Filter Banks: Polyphase Domain

- Decimation and upsampling: 2-channel example
  - Gathering common branches:



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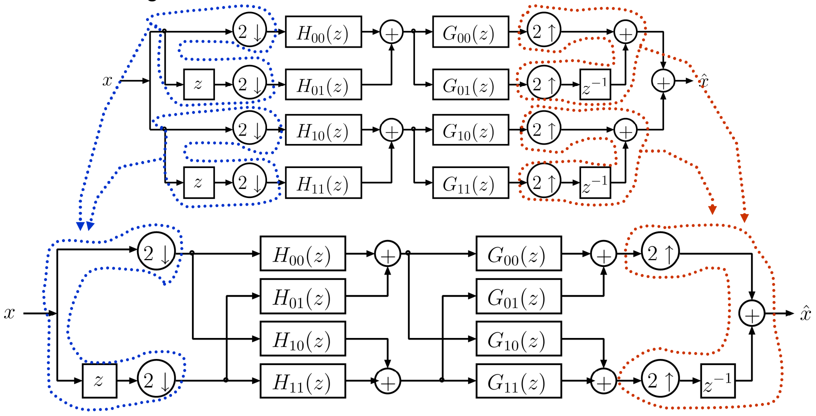
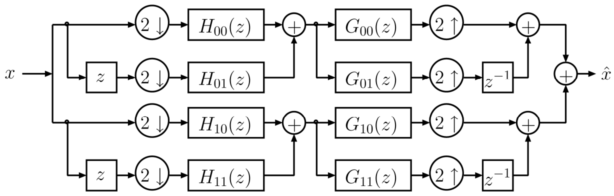
# Analysis of Filter Banks: Polyphase Domain

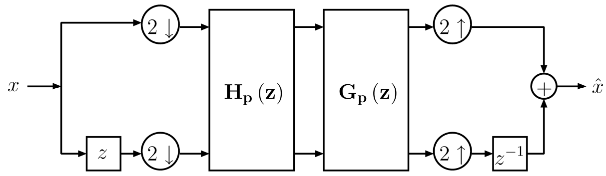
- Decimation and upsampling: 2-channel example
  - Gathering common branches:



# Analysis of Filter Banks: Polyphase Domain

- Decimation and upsampling: 2-channel example
  - Gathering common branches:







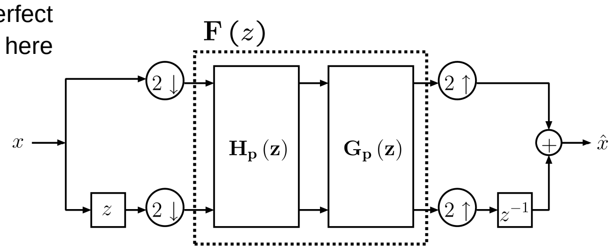

# Analysis of Filter Banks: Polyphase Domain

- What's special about the *"Polyphase-Domain"*?
  - We know what ***aliasing free*** polyphase transfer functions must look like:

**pseudo-circulant transfer function**

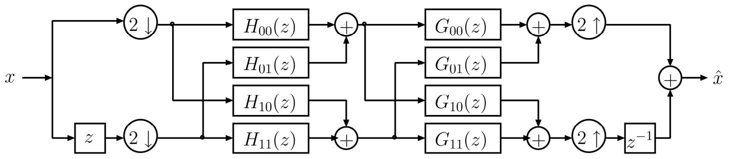
$$\mathbf{F}(z) = \begin{pmatrix} F_0(z) & F_1(z) & \dots & F_{N-1}(z) \\ zF_1(z) & \ddots & \dots & F_{N-2}(z) \\ \ddots & \ddots & \ddots & \vdots \\ zF_{N-1}(z) & \dots & zF_1(z) & F_0(z) \end{pmatrix}$$

No need for perfect reconstruction here



# Analysis of Filter Banks: Polyphase Domain

- Decimation and upsampling: 2-channel example
  - Analysis and Synthesis: ( $z^2$ is used in the full sampling rate domain)




$$\mathbf{Y}(z^2) = \underbrace{\begin{pmatrix} H_{00}(z^2) & H_{01}(z^2) \\ H_{10}(z^2) & H_{11}(z^2) \end{pmatrix}}_{\mathbf{H}_\mathrm{p}(z)} \begin{pmatrix} 1 \\ z \end{pmatrix} X(z)$$

$$\hat{X}(z) = \begin{pmatrix} 1 & z^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} G_{00}(z^2) & G_{10}(z^2) \\ G_{01}(z^2) & G_{11}(z^2) \end{pmatrix}}_{\mathbf{G}_\mathrm{p}(z)} \mathbf{Y}(z^2)$$





# Analysis of Filter Banks: Polyphase Domain

- Perfect reconstruction: 2-channel example

$$\begin{pmatrix} 1 & z^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} G_{00}(z^2) & G_{10}(z^2) \\ G_{01}(z^2) & G_{11}(z^2) \end{pmatrix}}_{\mathbf{G}_\mathrm{p}(z)} \underbrace{\begin{pmatrix} H_{00}(z^2) & H_{01}(z^2) \\ H_{10}(z^2) & H_{11}(z^2) \end{pmatrix}}_{\mathbf{H}_\mathrm{p}(z)} \begin{pmatrix} 1 \\ z \end{pmatrix} = \mathbf{I}$$

$$\mathbf{G}_\mathrm{p}(z)\mathbf{H}_\mathrm{p}(z) = \mathbf{I}$$

- Orthonormality:

# Analysis of Filter Banks: Polyphase Domain

- Perfect reconstruction: 2-channel example

$$\begin{pmatrix}1 & z^{-1}\end{pmatrix}\underbrace{\begin{pmatrix}G_{00}(z^2) & G_{10}(z^2)\\ G_{01}(z^2) & G_{11}(z^2)\end{pmatrix}}_{\mathbf{G_p}(z)}\underbrace{\begin{pmatrix}H_{00}(z^2) & H_{01}(z^2)\\ H_{10}(z^2) & H_{11}(z^2)\end{pmatrix}}_{\mathbf{H_p}(z)}\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix} = \mathbf{I}$$

$$\mathbf{G_p}(z)\mathbf{H_p}(z) = \mathbf{I}$$

- Orthonormality:
  - Analysis filters are time reversed synthesis filters

$$\mathbf{H_p}(z) = \mathbf{G_p^T}(z^{-1}) \qquad \mathbf{G_p}(z)\mathbf{G_p^T}(z^{-1}) = \mathbf{I}$$

$\{\}^{\mathbf{T}}$ is the hermitian transpose



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# Analysis of Filter Banks: Polyphase Domain

- Perfect reconstruction: 2-channel example

$$\begin{pmatrix}1 & z^{-1}\end{pmatrix}\underbrace{\begin{pmatrix}G_{00}(z^2) & G_{10}(z^2)\\ G_{01}(z^2) & G_{11}(z^2)\end{pmatrix}}_{\mathbf{G_p}(z)}\underbrace{\begin{pmatrix}H_{00}(z^2) & H_{01}(z^2)\\ H_{10}(z^2) & H_{11}(z^2)\end{pmatrix}}_{\mathbf{H_p}(z)}\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix} = \mathbf{I}$$

$$\mathbf{G_p}(z)\mathbf{H_p}(z) = \mathbf{I}$$

- Orthonormality:
  - Analysis filters are time reversed synthesis filters

$$\mathbf{H_p}(z) = \mathbf{G_p^T}(z^{-1}) \qquad \mathbf{G_p}(z)\mathbf{G_p^T}(z^{-1}) = \mathbf{I}$$

- Alias free:

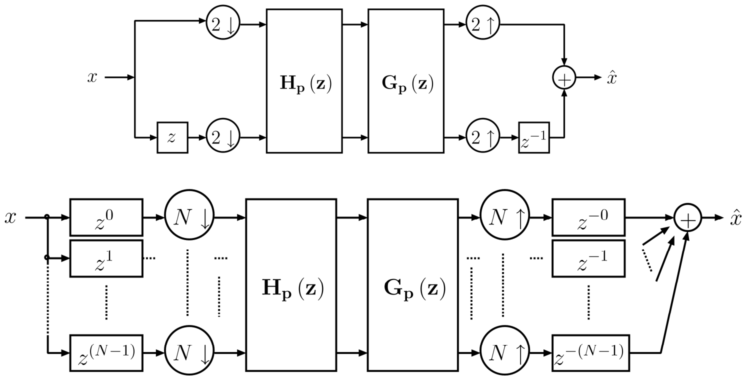
  $\mathbf{G_p}(z)\mathbf{H_p}(z)$ pseudo-circulant

  or $det(\mathbf{H_p}(z)) \neq 0$, i.e. $\mathbf{H_p}(z)$ full rank

$\{\}^{\mathbf{T}}$ is the hermitian transpose

# Analysis of Filter Banks: Polyphase Domain

- The results from the 2-channel case can be generalized to N-channel filter banks








# Relations between Modulation & Polyphase Domain

- Analysis

$$\underbrace{\begin{pmatrix} H_{00}(z^2) & H_{01}(z^2) \\ H_{10}(z^2) & H_{11}(z^2) \end{pmatrix}}_{H_p(z^2)} = \frac{1}{2} \underbrace{\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix}}_{H_m(z)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}$$

- Synthesis

$$\underbrace{\begin{pmatrix} G_{00}(z^2) & G_{01}(z^2) \\ G_{10}(z^2) & G_{11}(z^2) \end{pmatrix}}_{G_p(z^2)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} G_0(z) & G_0(-z) \\ G_1(z) & G_1(-z) \end{pmatrix}}_{G_m(z)}$$

## Relations between Time & Polyphase Domain

- Consider the time- domain synthesis matrix in the frequency domain

$$\mathbf{T}_s(z) = \sum_{i=0}^{K-1} \mathbf{S}_i z^{-i} \qquad \mathbf{S}_i = \begin{pmatrix} g_0[2i] & g_1[2i] \\ g_0[2i+1] & g_1[2i+1] \end{pmatrix}$$

$$\boxed{\mathbf{T}_s(z) = \mathbf{G}_p(z)}$$

- The same for the analysis matrix

$$\mathbf{T}_a(z) = \sum_{i=0}^{K-1} \mathbf{A}_i z^{-i} \quad \mathbf{A}_i = \begin{pmatrix} h_0[2(K-i)-1] & h_0[2(K-i)-2] \\ h_1[2(K-i)-1] & h_1[2(K-i)-2] \end{pmatrix}$$

$$\boxed{\mathbf{T}_a(z) = z^{-K+1}\mathbf{H}_p(z^{-1}) \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix}}$$





## Outline

- Introduction
  - Orthonormality
  - Biorthogonality
- Orthonormal expansions and filter banks
  - Haar expansion
  - Sinc expansion
- Analysis of filter banks
  - Time domain
  - Modulation domain
  - Polyphase domain
  - Relations between time, modulation, and polyphase domain
- **Results on filter banks**
  - **Biorthogonal Relations**
  - **Orthogonal Relations**

## Reconstruction

- Alias free reconstruction
- Perfect reconstruction
  - Filter bank output is a possibly scaled and delayed version of the input

$$\hat{X}(z) = cz^{-k}X(z)$$





## Alias- free Reconstruction

- Polyphase domain
  - Transfer matrix $T_P$ is pseudocirculant

$$F_{ij}(z) = \begin{cases} F_{0,j-i}(z) & j \geq i, \\ zF_{0,N+j-i}(z) & j < i. \end{cases}$$

  - 2 channel case

$$F(z) = \begin{pmatrix} F_0(z) & F_1(z) \\ zF_1(z) & F_0(z) \end{pmatrix}$$

- Polyphase analysis filters
  - Determinant of $H_P(z)$ is not identically zero, so that $H_P(z)$ has full rank

# Perfect Reconstruction

- FIR filter
  - For a critically sampled FIR analysis filter bank, perfect reconstruction with FIR filter is possible *if and only if* **det($\mathbf{H_p}$(z)) is a pure delay**.
  - Cosine modulated filter banks
    - All filters are calculated from one L=2N length prototype low-pass filter $h_{pr}[n]$ by modulation $\left[-\frac{\pi}{2N}, \frac{\pi}{2N}\right]$
    - For perfect reconstruction $h_{pr}^2[i] + h_{pr}^2[N-1-i] = 2$ (power complementary)
    - Cosine modulated filters form the orthonormal base:

$$h_k[i] = \frac{1}{\sqrt{N}} h_{pr}[n] \cdot \cos\left(\frac{2k+1}{4N}(2n-N+1)\pi\right)$$



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# Summary of Biorthongonality Relations

These statements are equivalent

1) $< h_i[-n], g_j[n-Nm] >= \delta[i-j]\delta[m]$

2) $\mathbf{T}_s \cdot \mathbf{T}_a = \mathbf{T}_a \cdot \mathbf{T}_s = \mathbf{I}$

3) $\frac{1}{N}\mathbf{G}_m(z)\mathbf{H}_m(z) = \frac{1}{N}\mathbf{H}_m(z)\mathbf{G}_m(z) = \mathbf{I}$

4) $\mathbf{G}_p(z)\mathbf{H}_p(z) = \mathbf{H}_p(z)\mathbf{G}_p(z) = \mathbf{I}$

**Biorthogonality is equal to perfect reconstruction**

## Summary of Orthonormality Relations

These statements are equivalent

1) $< g_i[n], g_j[n+Nm] >= \delta[i-j]\delta[m]$

2) $\mathbf{T}_s^T \cdot \mathbf{T}_s = \mathbf{T}_s \cdot \mathbf{T}_s^T = \mathbf{I}$

3) $\frac{1}{N}\mathbf{G}_m^T(z^{-1})\mathbf{G}_m(z) = \frac{1}{N}\mathbf{G}_m^T(z^{-1})\mathbf{G}_m(z) = \mathbf{I}$

4) $\mathbf{G}_p^T(z^{-1})\mathbf{G}_p(z) = \mathbf{G}_p(z)\mathbf{G}_p^T(z^{-1}) = \mathbf{I}$

$\mathbf{T}_a = \mathbf{T}_s^T$

$\mathbf{H}_m(z) = \mathbf{G}_m^T(z^{-1})$

$\mathbf{H}_p(z) = \mathbf{G}_p^T(z^{-1})$





## Main Reference

M. Vetterli and J. Kovacevic:
**Wavelets and subband coding**
Prentice Hall, 1995.

# Thank you for your attention!

Please feel free to ask questions.