Fundamentals of Multirate Systems

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Outline

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- **Interconnection of Building Blocks**
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Introduction

- Multirate Systems
  - Systems that employ more than one sampling rate
  - Result in more efficient processing of signals
  - Sampling rates at various internal points can be kept as small as possible
  - Also results in “aliasing”, that can be cancelled

Introduction

- Sampling rate alteration can be performed by using "Decimators" and "Expanders".
- Multirate Systems have applications in
  - Digital Audio Systems
  - Subband Coding of Speech and Image Signals
  - Adaptive Filters
  - Digital Telephony

Basic Multirate Operations

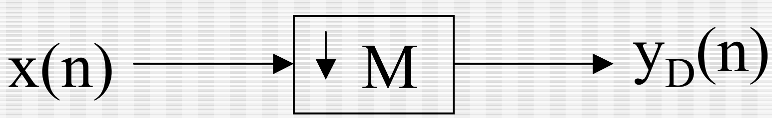
- The most basic operations in multirate digital signal processing are
  - Decimation
  - Interpolation
- These operations can be performed by the building blocks known as
  - Decimator
  - Expander

M-fold Decimator

- It takes an input sequence *x(n)* and produces the output sequence

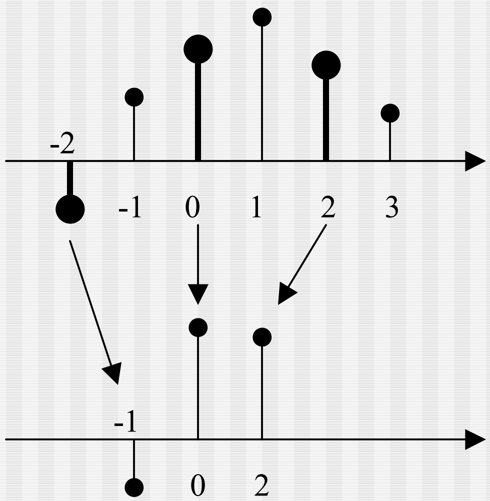
$$y_D(n) = x(Mn)$$

  where *M* is an integer
- Retains only those samples of *x(n)* which occur at time equal to multiples of *M*.



M-fold Decimator

- For example, M = 2



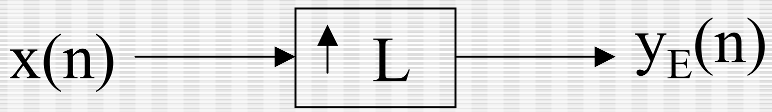
- Decimator is also called a *downsampler*, *subsampler* or a *compressor*

L-fold Expander

- This device takes an input *x(n)* and produces an output sequence

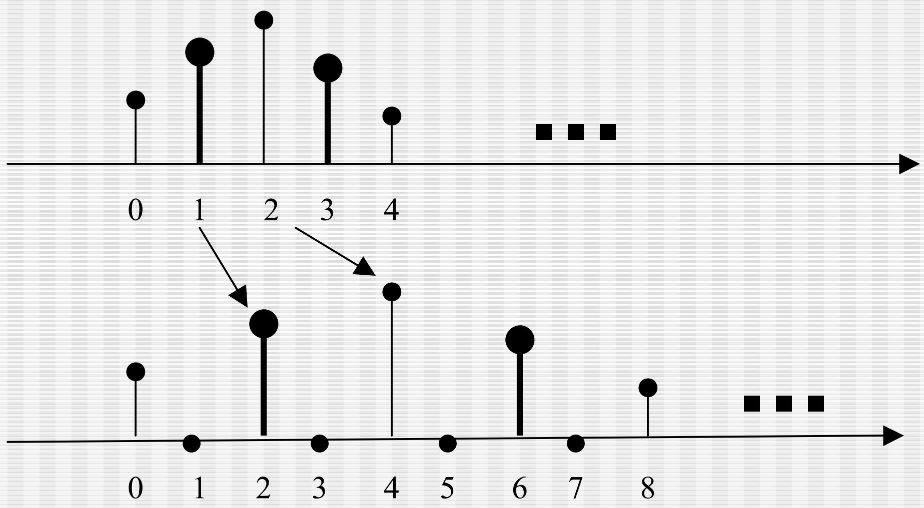
$$y_E(n) = \begin{cases} x(n/L), & \text{if } n \text{ is integer-multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

Here *L* is an integer



L-fold Expander

- For example, L = 2



- Other names for *expander* are *upsampler* and *interpolator*

Transform Domain Analysis

- Expander

$$Y_E(z) = \sum_{n=-\infty}^{\infty} y_E(n) z^{-n} = \sum_{n=mul.ofL} y_E(n) z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} y_E(kL) z^{-kL} = \sum_{k=-\infty}^{\infty} x(k) z^{-kL}$$
$$= X(z^L)$$

So

$$Y_E(e^{jw}) = X(e^{jwL})$$

- This means that $Y_E(e^{j\omega})$ is an L-fold compressed version of $X(e^{j\omega})$.
- The expander creates an imaging effect

Transform Domain Analysis

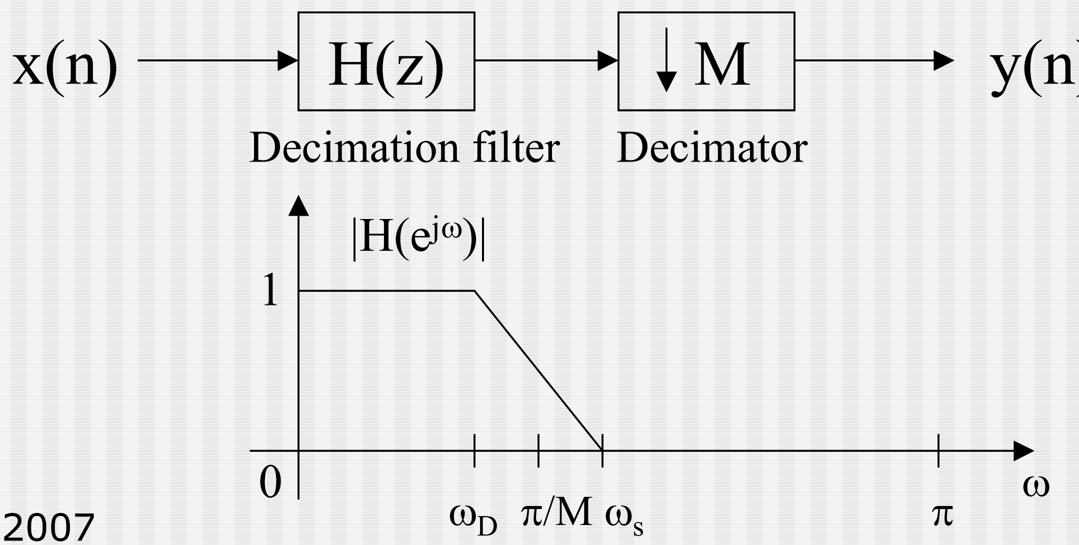
- Decimator

$$Y_D(e^{jw}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(w-2\pi k)/M})$$

- Decimation produces expansion in frequency domain giving rise to "aliasing"
- Aliasing can be avoided if *x(n)* is a lowpass signal bandlimited to the region $|\omega| < \pi/M$

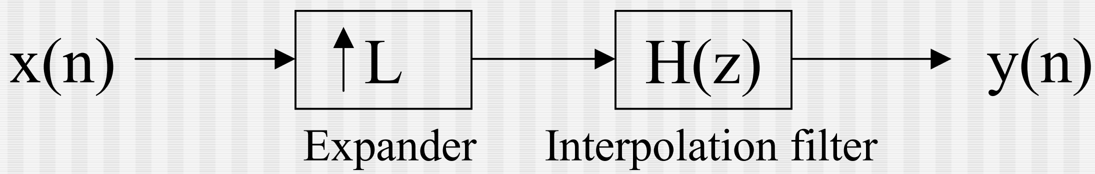
Decimation Filter

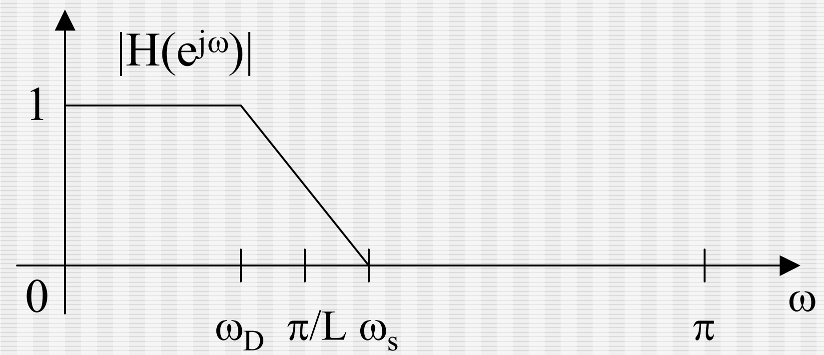
- In most applications, the decimator is preceded by a lowpass digital filter called the *decimation filter*.
- This filter ensures that the signal being decimated is bandlimited.



Interpolation Filter

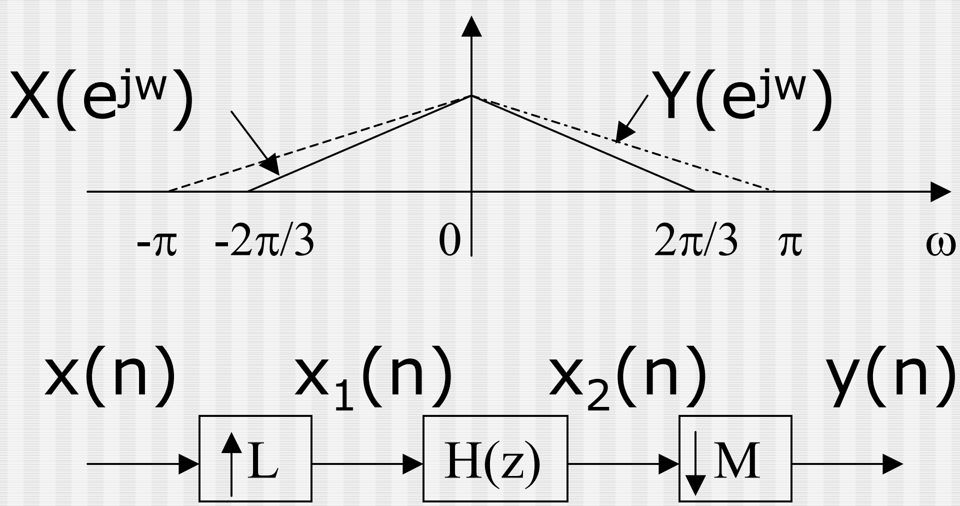
- A digital filter that *follows* an expander.
- Used to suppress all the images
- Typically it is lowpass with cutoff frequency π/L.





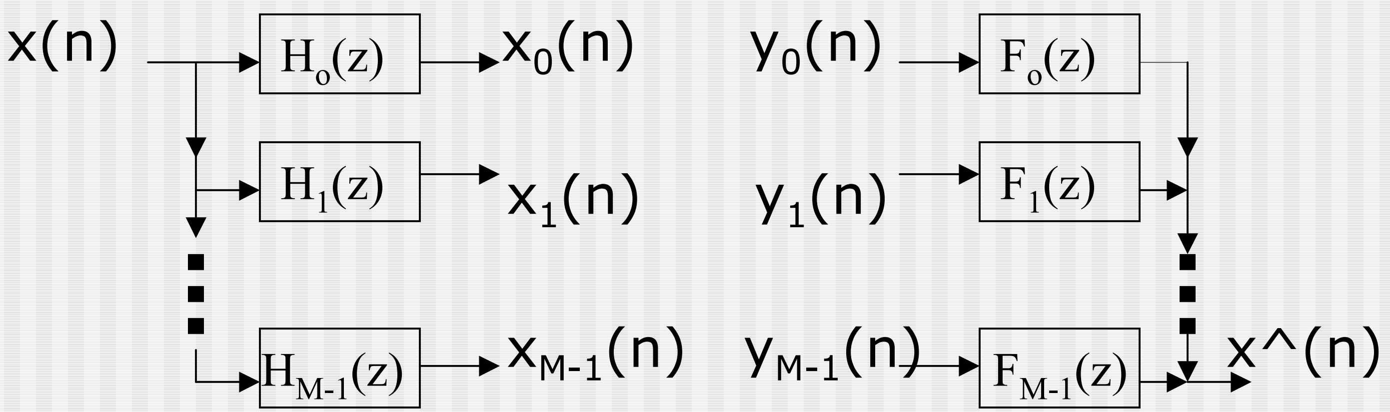
Fractional Sampling Rate Alteration

- In some applications it is necessary to change the rate by a rational fraction (such as *L/M* or *M/L*) .
- Example: L=2, M=3, M/L=1.5



Digital Filter Banks

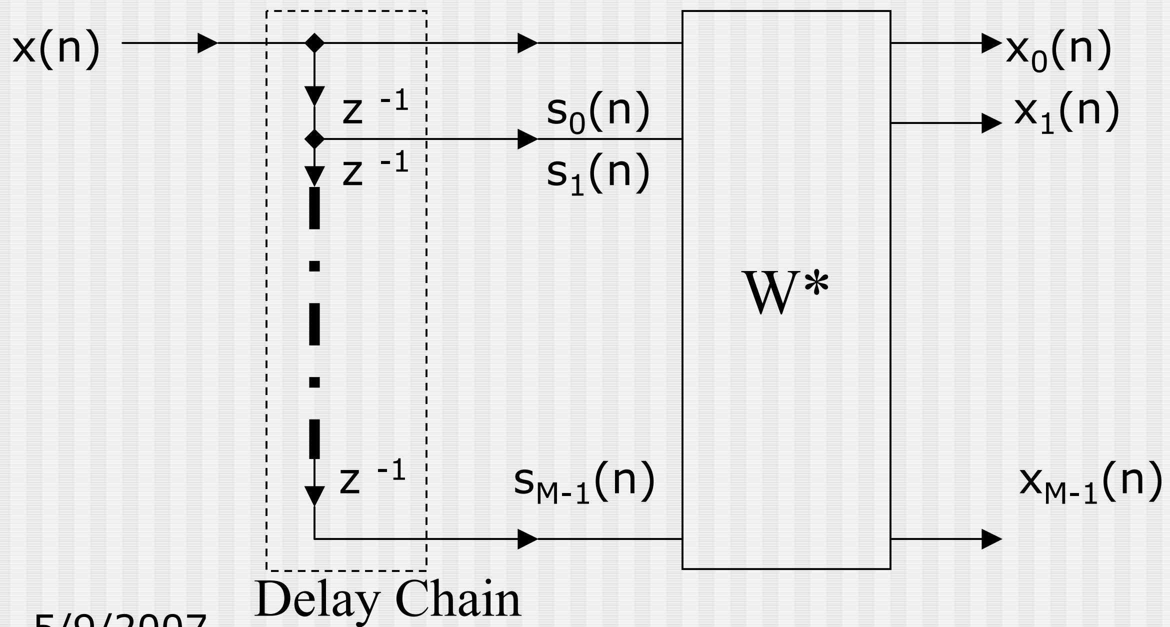
- Collection of digital filters, with a common input or a common output. So there can be two types of filter banks
  - Analysis Bank – Composed of Analysis Filters
  - Synthesis Bank – Composed of Synthesis Filters



Digital Filter Banks

- <u>Example</u>: *The DFT Filter Bank*

  Consider a filter bank based on DFT matrix



Example: *The DFT Filter Bank* *(contd.)*

W is an $M \times M$ DFT matrix with elements $[W]_{km} = W^{km}$, where $W = e^{-j2\pi/M}$.

$$x_i(n) = \sum_{k=0}^{M-1} s_i(n) W^{-ki}$$

$$X_k(z) = \sum_{i=0}^{M-1} S_i(z) W^{-ki}$$

$$= \sum_{i=0}^{M-1} z^{-i} W^{-ki} X(z) = \sum_{i=0}^{M-1} (zW^k)^{-i} X(z)$$

So $X_k(z) = H_k(z) X(z)$ where

with

$$H_k(z) \underset{=}{\Delta} H_0(zW^k)$$

$$H_0(z) = 1 + z^{-1} + \ldots + z^{-(M-1)}$$

The system is equivalent to analysis bank with analysis filters $H_k(z)$

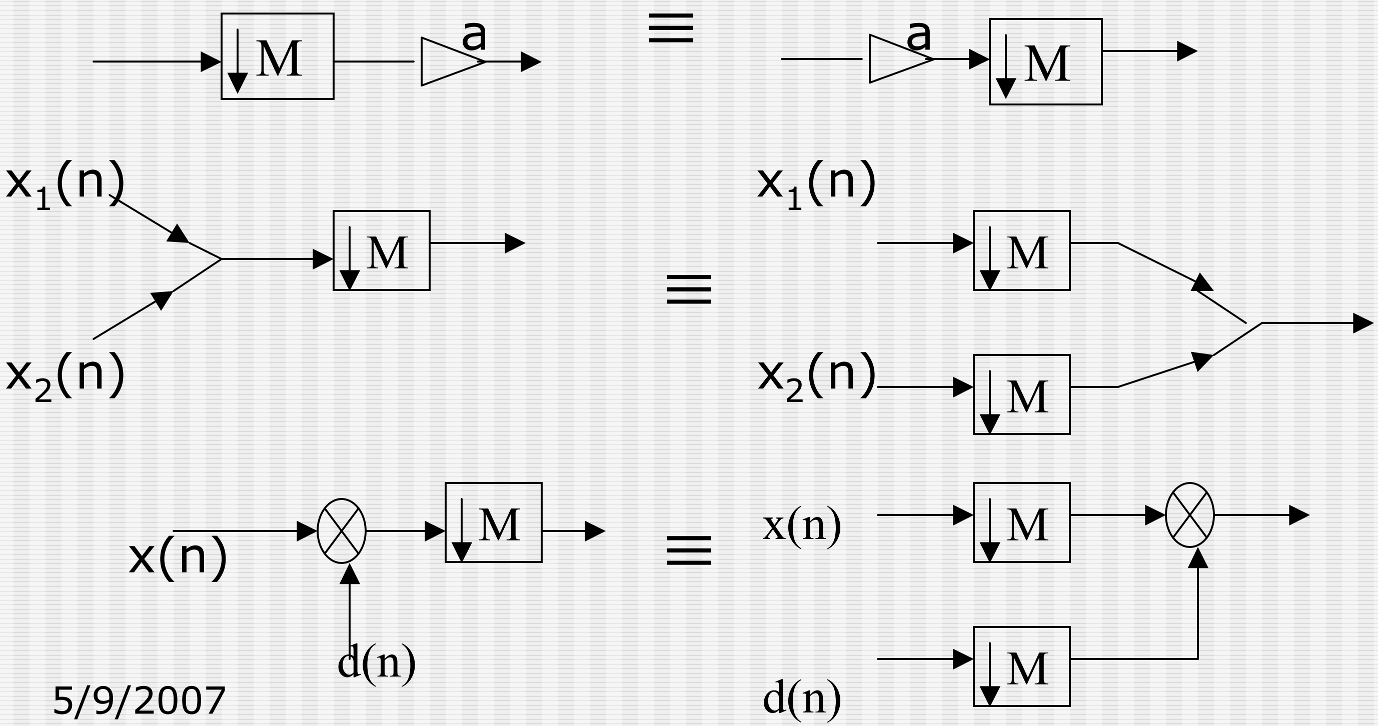
Time Domain Descriptions of Multirate Filters

- The input-output relation in the time domain for decimation, interpolation and fractional decimation filters is given by

$$y(n) = \begin{cases} \sum_{k=-\infty}^{\infty} x(k)h(nM-k) & , \quad M\text{-fold decimation filter} \\ \sum_{k=-\infty}^{\infty} x(k)h(n-kL) & , \quad L\text{-fold interpolation filter} \\ \sum_{k=-\infty}^{\infty} x(k)h(nM-kL) & , \quad M/L\text{-fold decimation filter} \end{cases}$$

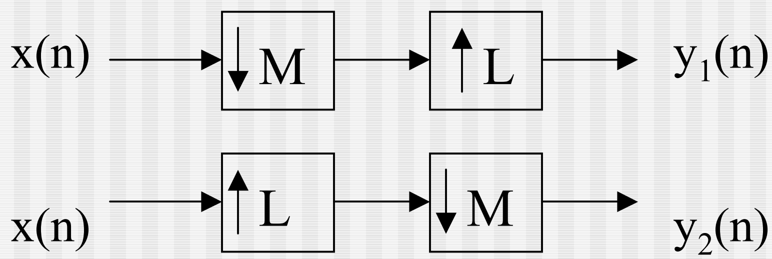
Interconnection of Building Blocks

- Some commonly occurring building blocks in multirate systems



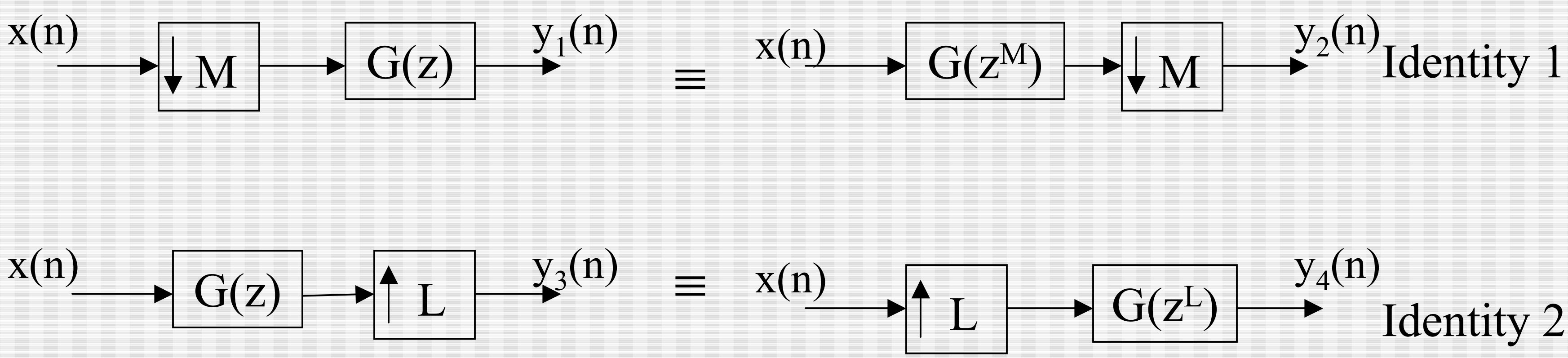
Decimator-expander cascades

- The two structures are equivalent [i.e $y_1(n)=y_2(n)$ for every possible input $x(n)$] iff *L* and *M* are relatively prime integers (i.e. greatest common divisor = 1)



Noble Identities

- Noble identities are very useful in the theory and implementations of multirate systems.



Polyphase Representation

- This representation permits great simplification of theoretical results
- Leads to computationally efficient implementations of decimation and interpolation filters as well as filter banks.

Polyphase Representation

- *Basic Idea*

  Consider a filter $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$

  By separating the even numbered coefficients of *h(n)* from odd numbered one,

$$H(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h(2n+1) z^{-2n}$$

Defining

$$E_0(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-n}, E_1(z) = \sum_{n=-\infty}^{\infty} h(2n+1) z^{-n}$$

Therefore

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

Polyphase Representation

- *Basic Idea (contd.)*
  Suppose an integer *M*, then *H(z)* can be decomposed as

$$H(z)=\sum_{n=-\infty}^{\infty}h(nM)z^{-nM}$$

$$+z^{-1}\sum_{n=-\infty}^{\infty}h(nM+1)z^{-nM}$$

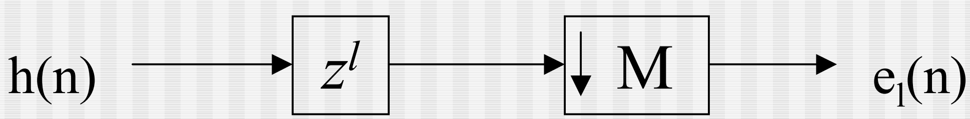
$$\vdots$$

$$z^{-(M-1)}\sum_{n=-\infty}^{\infty}h(nM+M-1)z^{-nM}$$

This can be compactly written as $H(z)=\sum_{l=0}^{M-1}z^{-l}E_l(z^M)$

$$E_l(z)=\sum_{n=-\infty}^{\infty}e_l(n)z^{-n}$$ (Type 1 polyphase)

where $e_l(n)\triangleq h(Mn+l)$ $0\le l\le M-1$



Polyphase Representation

$H(z)$ can also be written as

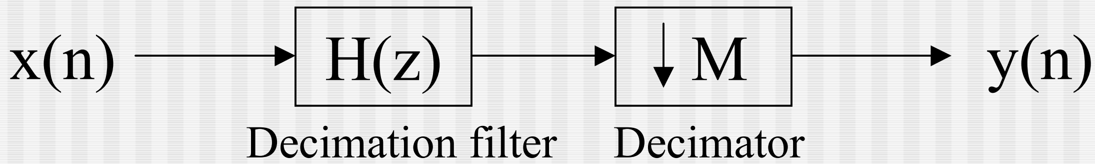
$$H(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_l(z^M) \quad \text{(Type 2 polyphase)}$$

The Type 2 polyphase components $R_l(z)$ are permutations of $E_l(z)$, that is

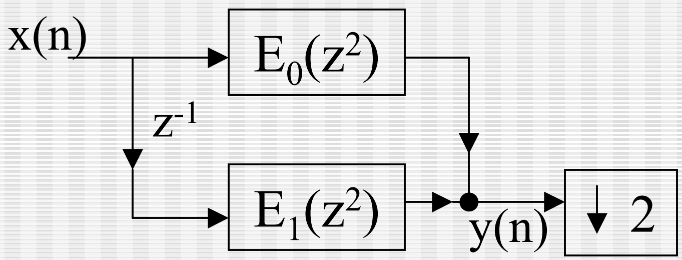
$$R_l(z) = E_{M-1-l}(z)$$

Efficient Structure for Decimation Filters

- Consider the decimation circuit

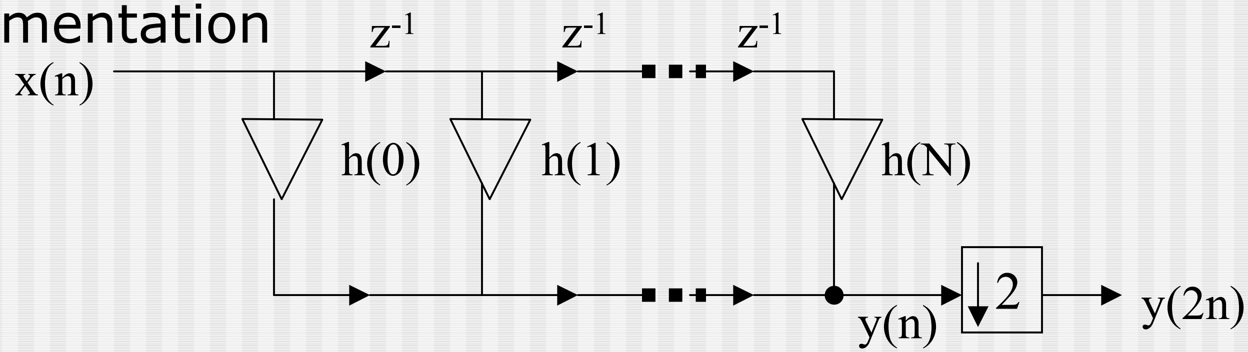


- We can represent this circuit using polyphase implementation for M=2 as



Efficient Structure for Decimation Filters

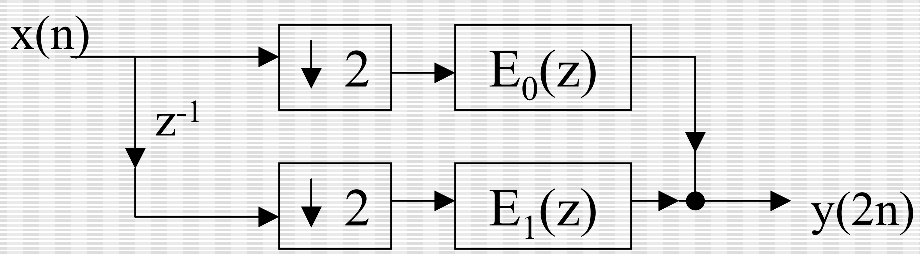
- Consider a *N*th order FIR, with traditional direct implementation



- Computes only even numbered samples *y(*2*n)* requiring N+1 multiplications and N additions.
- Time change from 2n to 2n+1, change the stored signals in the delay
- Computation must be completed in one unit of time.
- Inefficient resource utilization

Efficient Structure for Decimation Filters

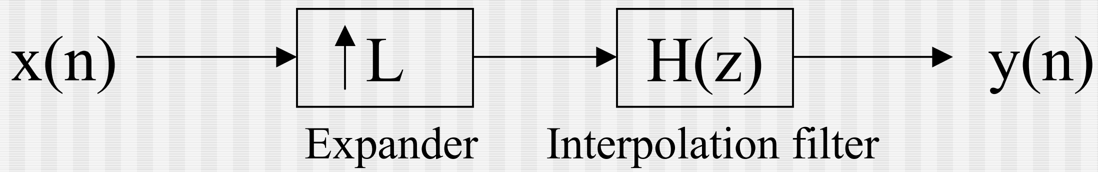
- Consider the polyphase implementation for H(z)



- Let n_0 and n_1 be the orders of $E_0(z)$ and $E_1(z)$ ($N+1 = n_0+n_1+2$)
- $E_l(z)$ requires n_l+1 multiplications and n_l additions. Total cost is again N+1 multipliers and N adders.
- Rate of operation for $E_l(z)$ is $(N+1)/2$ MPUs and $N/2$ APUs.
- Efficient resource utilization.

Efficient Structure for Interpolation Filters

- Consider and interpolation circuit

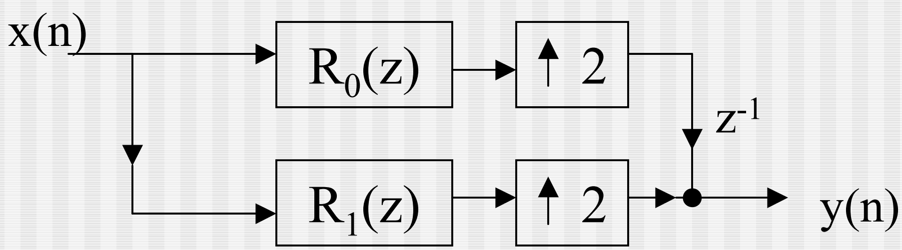


- In direct implementation H(z) performs inefficiently, due to at most 50% nonzero coefficients.
- Only 50% multipliers *h(n)* have nonzero input, and job must be completed in *half unit of time.*
- Inefficient resource utilization

Efficient Structure for Interpolation Filters

- Efficient structure can be obtained by using Type 2 polyphase decomposition.

$$H(z) = R_1(z^2) + z^{-1} R_0(z^2)$$



- $R_l(z)$ operates at input rate and each multiplier gets one unit of time to complete its task
- Complexity is $N + 1$ MPUs and $N - 1$ APUs, last adder after expander only performs interlacing

Polyphase Structure of DFT Filter Banks

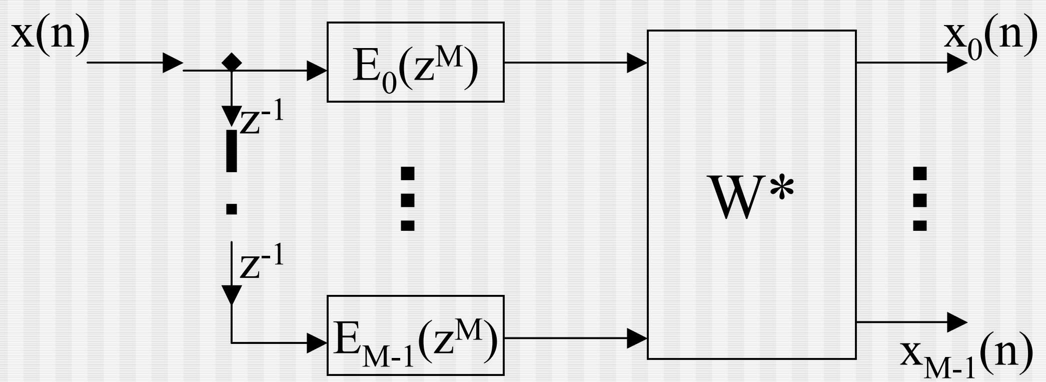
- Polyphase implementation of k-th filter in a uniform DFT bank is

$$H_k(z) = H_0(zW^k) = \sum_{l=0}^{M-1} (z^{-1}W^{-k})^l E_l(z^M)$$

With $X_k(z)$ denoting the output of $H_k(z)$

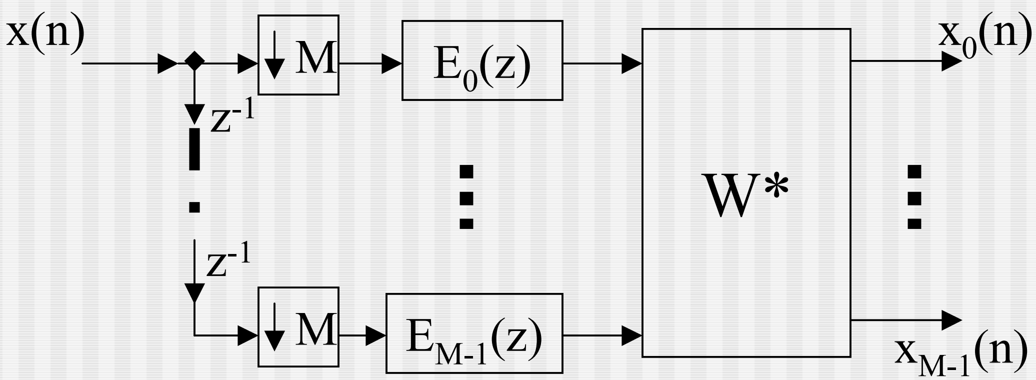
$$X_k(z) = \sum_{l=0}^{M-1} W^{-kl} (z^{-l} E_l(z^M) X(z))$$

This can be represented by



Decimated Uniform Filter Banks

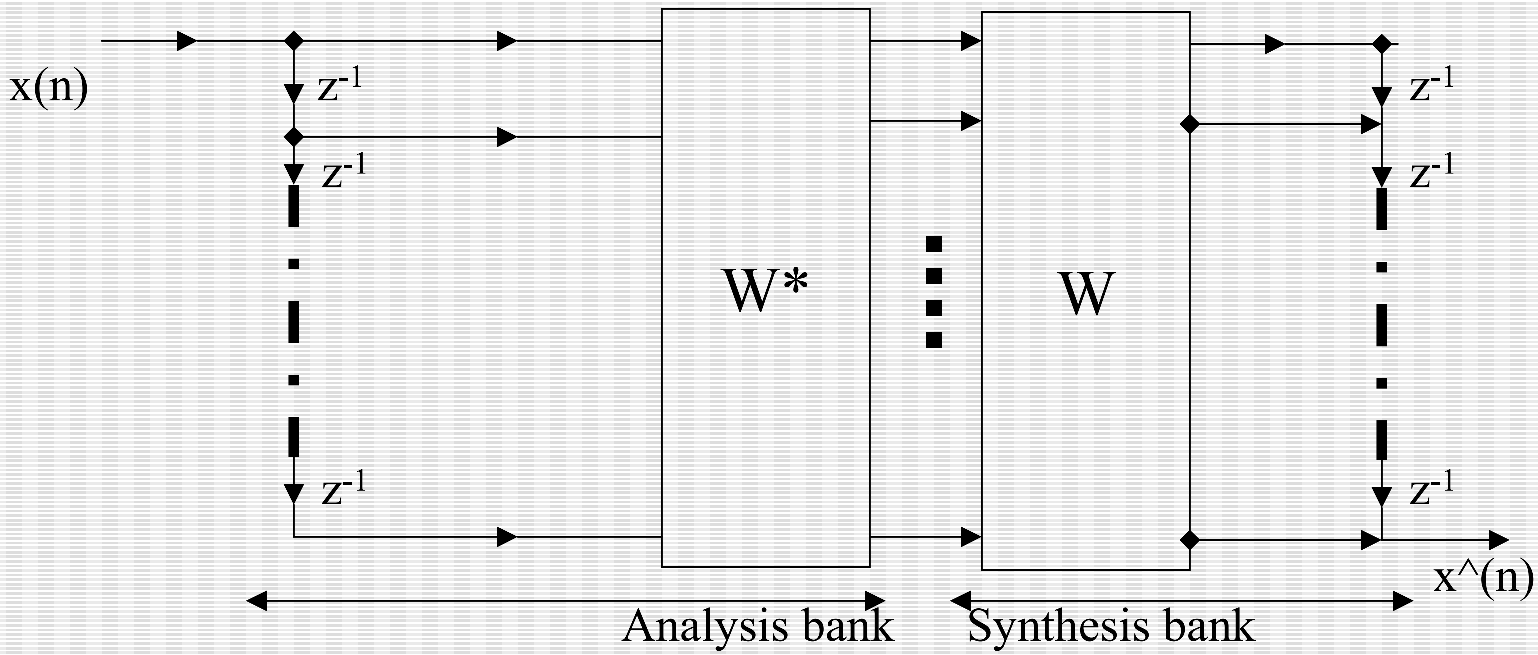
- In many applications such as QMF banks, outputs of $H_k(z)$ are decimated by M.



- This structure requires *M* times fewer MPUs and APUs.

Perfect Reconstruction (PR) Systems

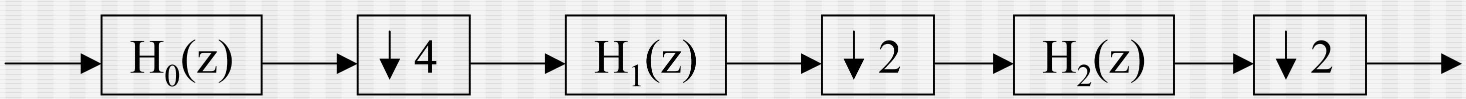
- A system in which $\hat{x}(n) = cx(n-n_0)$ for some $c \neq 0$ and integer n_0 .



Multistage Implementations

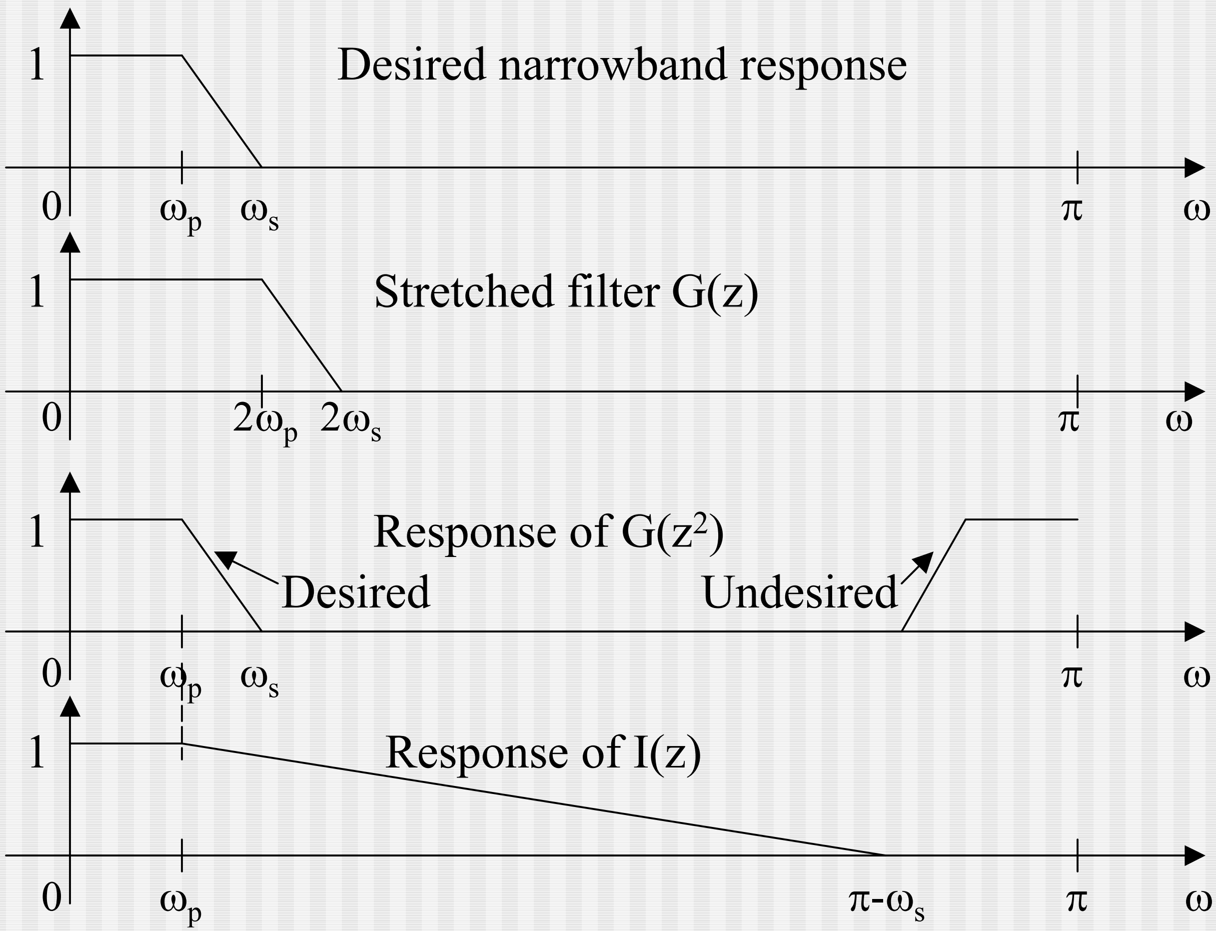
- Used in applications, where decimation or interpolation by a large factor is required.
- Results in more efficient systems

e.g. M =16. Since 16 = 4 × 2 × 2. So system can be implemented in three stages.

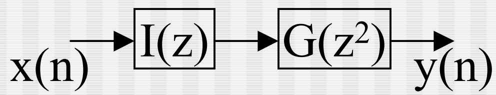


Interpolated FIR *(IFIR)* Approach

- Efficient technique for the design of narrowband lowpass filters.

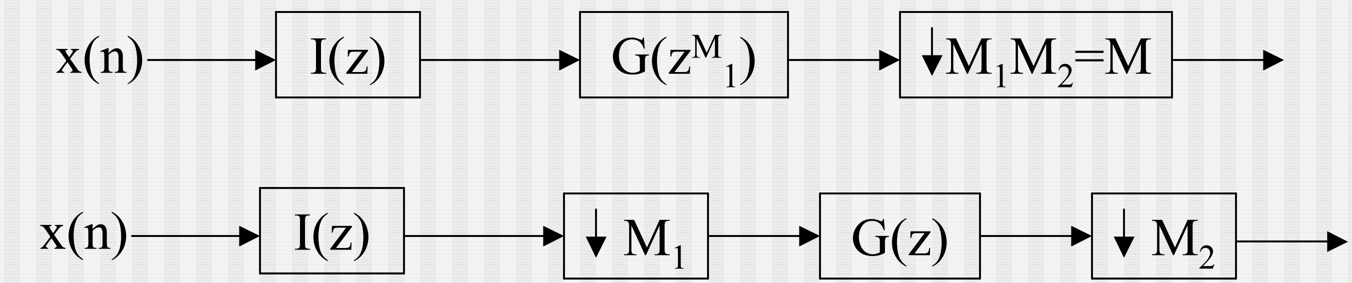


Interpolated FIR *(IFIR)* Approach

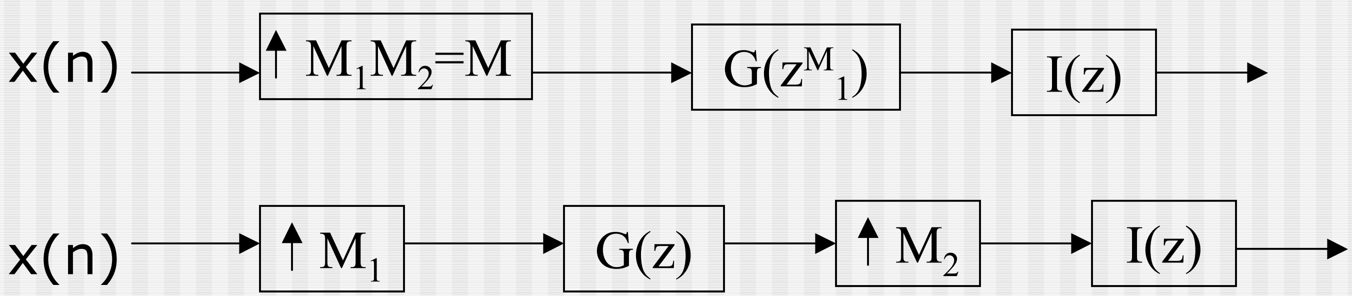


- If order of *G(z)* and *I(z)* are N_g and N_i , the system requires $(N_g+1)+(N_i+1)$ multiplications and $(N_g + N_i)$ additions.
- Filter *G(z)* is called the *model filter* and *I(z)* an *image suppressor.*
- It is also possible to stretch the specifications by an amount $M_1>2$, so $G(z^{M1})$ has M_1-1 unwanted passbands.

Multistage Design of Decimation and Interpolation Filters



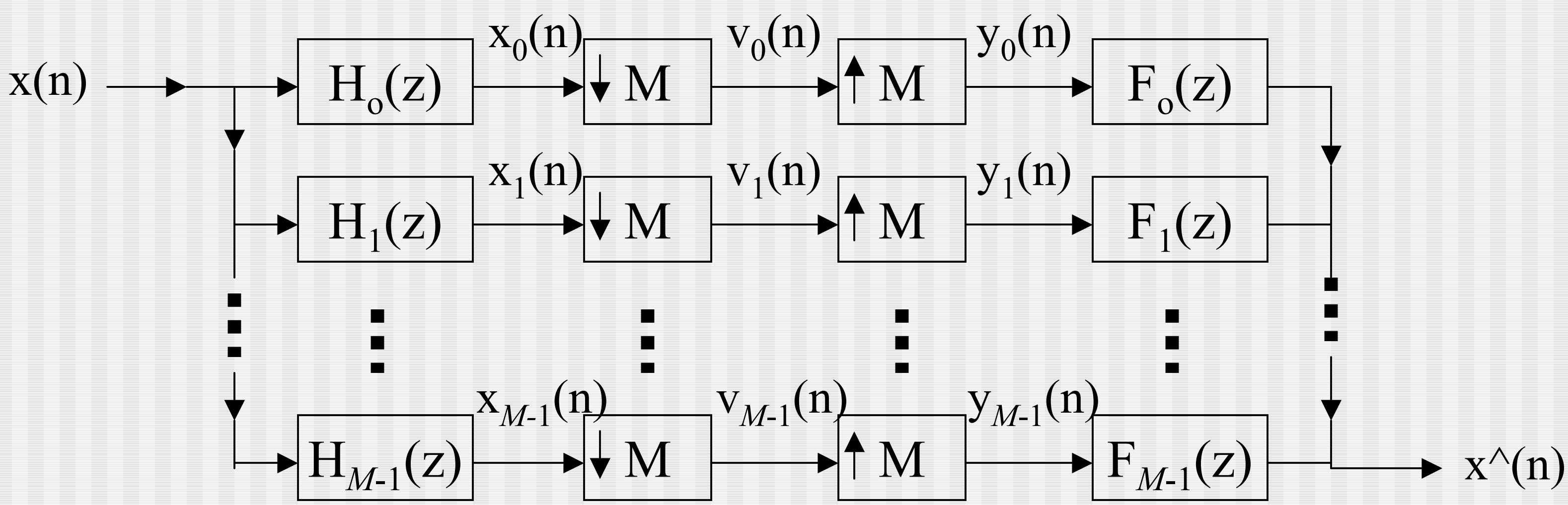
The two-stage decimator developed from IFIR decimation



The two-stage interpolator developed from IFIR interpolation

Applications of Multirate Systems

- *Subband Coding*
  - Split a signal into *M* subbands
  - Decimate each subband signal by *M*
  - Allocate bits for samples in each subband depending on the energy content.

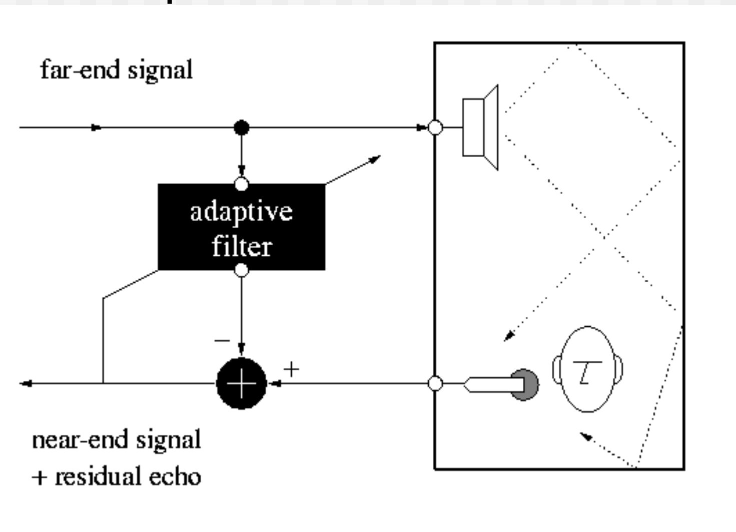


Applications of Multirate Systems

- *Subband Coding (contd)*
  - Examples of subband coding include
    - 'image coding (e.g. wavelet filter banks)'
    - 'audio coding' such as digital compact cassette (DCC), MiniDisc, MPEG etc.
  - General remarks on subband coding
    - For subband coding to work, knowledge about energy distribution of $X(e^{j\omega})$ is required
    - The filters $F_k(z)$ should be chosen carefully to cancel the *aliasing* introduced by band splitting and decimation

Filter Banks Applications

- *Subband adaptive filtering*
  - Example : Acoustic Echo Cancellation
    - Adaptive filter models (time-varying) acoustic echo path and produces a copy of the echo, which is then subtracted from microphone signal.
  - Difficult problem !
    - Long acoustic impulse responses
    - Time-varying

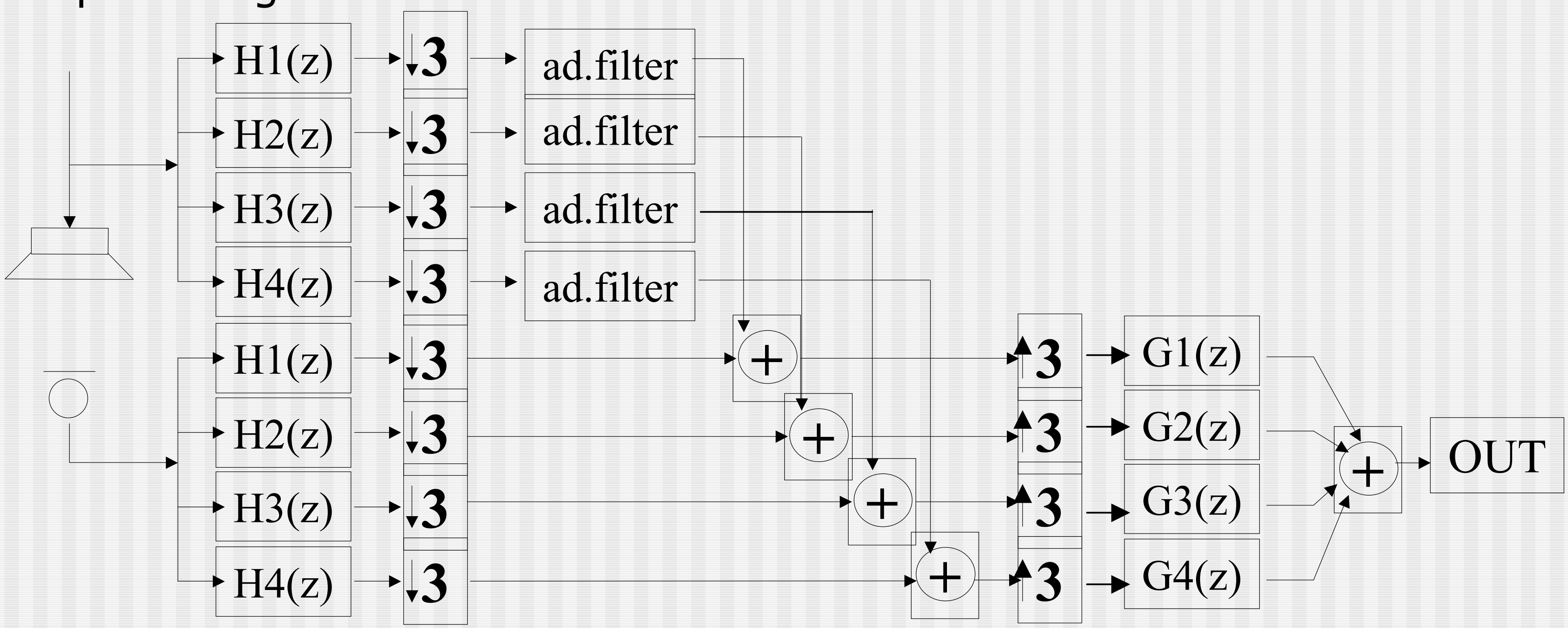


Filter Banks Applications

- *Subband Adaptive Filtering (contd)*

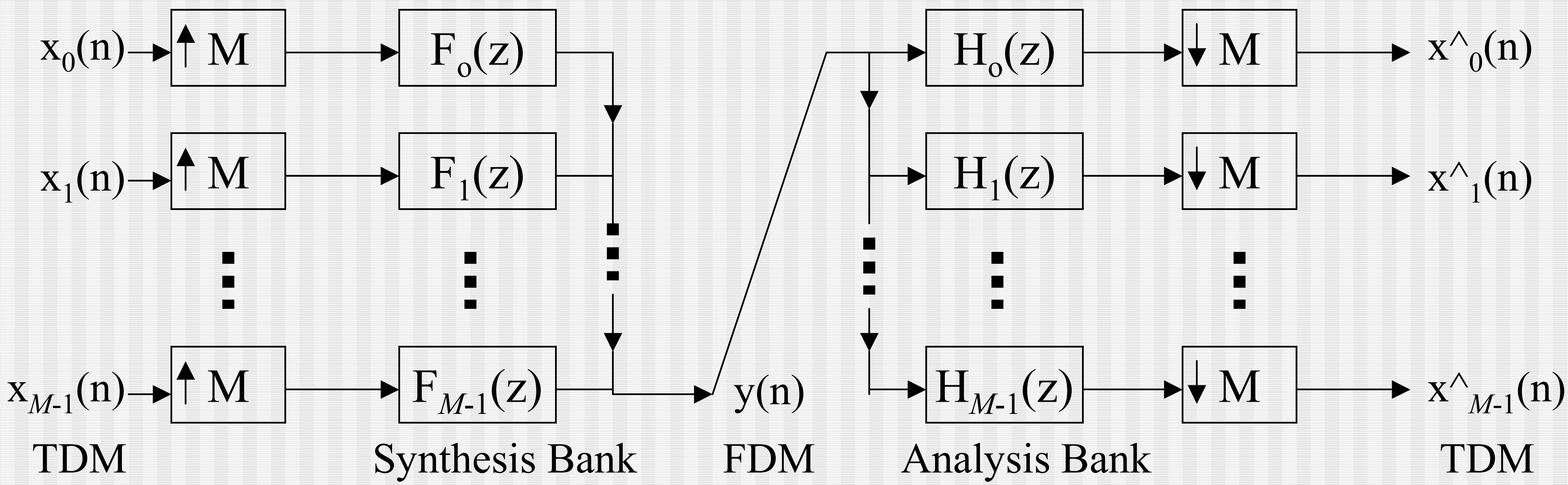
  - Subband filtering = M subband modeling problems instead of one fullband modeling problem

  - Perfect reconstruction guarantees distortion-free desired near-end speech signal



Filter Bank Applications

- *Transmultiplexers*
  - Time Division Multiplexed (TDM)
  - Frequency Division Multiplexed (FDM)



The complete transmultiplexer structure

Summary

- Basic building blocks of multirate systems and interconnection of these blocks has been presented
- An in depth analysis of Polyphase representation and their applications in multirate systems has been discussed.
- Applications of Multirate systems such as subband coding, subband adaptive filtering and transmux has been presented

References

- "Multirate Systems & Filter Banks", P.P. Vaidyanathan, Prentice Hall 1993.
- Digital Signal Processing II, Course by Marc Moonen, K.U.Leuven, ESAT/SISTA, Belgium, (*homes.esat.kuleuven.be/~moonen/*)