Fundamentals of Signal Decompositions

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Overview

- Vector Spaces, Hilbert Spaces, and Key Notions
- Basic concepts from Linear Algebra
- Fourier Theory and Sampling
- Time-Frequency Representations

Signal Decomposition

- Decompose signals into fundamental constituents
- Transformation from one domain to another (time - frequency)
- Simpler ways of analyzing and processing signals
- Complex operations are simplified (e.g. convolution, differentiation, integration)
- To understand later expansions mathematical background is needed...

Important questions

- Given a set of vectors $\{v_k\}$
  - Does $\{v_k\}$ span the space R^n or C^n?
  - Are the vectors linearly independent?
  - How can we find (orthonormal) bases for the space to be spanned?
  - Given a subspace of R^n or C^n and a general vector - finding an approximation in the least-squares sense?

Vector Spaces and Inner Products (1)

- Given vector space E
  - A subset M of E is a *subspace* of E if

    (a) $\forall$ x,y in M, x+y is in M
    (b) $\forall$ x in M, α in C or R, αx is in M

  - A subset of E is called *basis* of E when

    (a) E = span$(x_1, ..., x_n)$
    (b) $(x_1, ..., x_n)$ are lin. independent

    where

$$span(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| \alpha_i \in \mathrm{C} \text{ or } \mathrm{R}, x_i \in \mathrm{S} \right\}$$

$$lin.ind., \text{ if } \sum_{i=1}^{n} \alpha_i x_i = 0 \text{ is true only if}, \ \alpha_i = 0, \forall i$$

Vector Spaces and Inner Products (2)

- Inner Product on a vector space E over C(or R) is a complex-valued function defined on E x E mapping to a scalar with the following properties:
  (a) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
  (b) $\langle x, \alpha y\rangle = \alpha \langle x, y\rangle$
  (c) $\langle x, y\rangle^* = \langle y, x\rangle$
  (d) $\langle x, x\rangle \geq 0$
- e.g. Standard Inner Products

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f^*(t) g(t) dt$$

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x^*[n] y[n]$$

- Definition of Norm from Inner Product

$$\|x\| = \sqrt{\langle x, x\rangle}$$

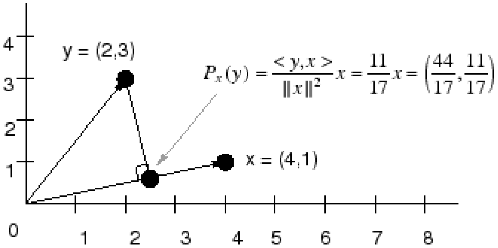
Vector Spaces and Inner Products (3)

- Given u,v $\in$ E

$$\text{orthogonal} \Leftrightarrow \langle u, v \rangle = 0$$

- Orthogonal projection

$$P_{x(y)} = \frac{\langle y, x \rangle}{\|x\|^2} x$$



Hilbert Space

- Vector space E with inner product is called *Inner Product Space*
- If every *Cauchy sequence* in E, converges to a vector in E, then E is *complete*
- A complete inner product space is called a ***Hilbert Space***
- e.g. Space of Square-Summable Sequences - Hilbert Space $l_2(Z)$,space of sequences x[n] having finite square sum or finite energy

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x^*[n] y[n]$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{n \in Z} |x[n]|^2}$$

Orthonormal Bases

- A set of vectors $S = \{x_i\}$ is said to be an orthonormal basis if
  - all x_i are orthogonal
  - all x_i are normalized to unit length
  - S is complete
- Orthonormal system $\{x_i\}$ is called orthonormal basis of E, if for every y in E,

$$y = \sum_k \alpha_k x_k \text{ ,where}$$
$$\alpha_k = \langle x_k, y \rangle$$

  coefficients of expansion are called *Fourier coefficients*
- Approximation is optimal in least-squares sense

General Bases

- Orthonormal bases are very convenient, but nonorthogonal or biorthogonal bases are important as well
- $\{x_i, x_i'\}$ constitutes a *pair of biorthogonal bases* iff

$$\left\langle x_i, x_j' \right\rangle = \delta[i-j] \text{ for all i,j in Z}$$

  Signal expansion formula becomes

$$y = \sum_k \langle x_k, y \rangle x_k' = \sum_k \langle x_k', y \rangle x_k$$

- Overcomplete Expansions
  - Signals as linear combination of an overcomplete set of vectors - no longer lin. ind.
  - expansion is not unique anymore

Eigenvectors, Eigenvalues

- a vector $p \neq 0$ is called *eigenvector* if

$$\mathbf{Ap} = \lambda \mathbf{p}$$

- roots of *characteristic polynomial* D(x) = det(x**I**-**A**) of matrix **A** are called *eigenvalues*
- if *nxn* matrix has n lin. ind. eigenvectors it can be *diagonalized*

$$A = T \Lambda T^{-1}$$

- importance of eigenvectors in study of linear operators comes from the following fact

  assuming vector x = $\sum \alpha_i v_i$

$$Ax = A\left(\sum_i \alpha_i v_i\right) = \sum_i \alpha_i \left(A v_i\right) = \sum_i \alpha_i \left(\lambda_i v_i\right)$$

Special Matrices (1)

circulant matrix

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & & & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{pmatrix}$$

Toeplitz matrix

$$T = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_{-n+1} \\ t_{-1} & t_0 & t_1 & \cdots & t_{-n+2} \\ t_{-2} & t_{-1} & t_0 & \cdots & t_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{-n+1} & t_{-n+2} & t_{-n+3} & \cdots & t_0 \end{pmatrix}$$

polynomial matrix

$$H = \begin{pmatrix} \sum a_i x^i & \cdots & \sum b_i x^i \\ \vdots & \ddots & \vdots \\ \sum c_i x^i & \cdots & \sum d_i x^i \end{pmatrix} = \sum_i H_i x^i$$

Special Matrices (2)

- An example for a matrix with toeplitz structure is the autocorrelation matrix

$$R_{xx} = \begin{pmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & \cdots \\ r_{xx}[-1] & r_{xx}[0] & r_{xx}[1] & \cdots \\ r_{xx}[-2] & r_{xx}[-1] & r_{xx}[0] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $r_{xx}[m] = E\{x[n]\,x[n+m]\}$

What we've done so far...

- ... Inner Product
- ... Hilbert Space
- ... Projection
- ... Orthonormal and general bases
- ... Eigenvectors and Eigenvalues

General Signal Expansions and Nomenclature

- Continous-time integral expansion (e.g. CTFT)

$$x(t) = \int X_\omega \Psi_\omega(t) d\omega \quad \text{with } X_\omega = \left\langle \tilde{\Psi}_\omega(t), x(t) \right\rangle$$

- Continous-time series expansion (e.g. CTFS)

$$x(t) = \sum_i X_i \Psi_i(t) \quad \text{with } X_i = \left\langle \tilde{\Psi}_i(t), x(t) \right\rangle$$

- Discrete-time integral expansion (e.g. DTFT)

$$x(t) = \int X_\omega \Psi_\omega[n] d\omega \quad \text{with } X_\omega = \left\langle \tilde{\Psi}_\omega[n], x[n] \right\rangle$$

- Discrete-time series expansion (e.g. DTFS)

$$x(t) = \sum_i X_i \Psi_i[n] \quad \text{with } X_i = \left\langle \tilde{\Psi}_i[n], x[n] \right\rangle$$

Continous-time Fourier Transform

- Fourier analysis formula

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \left\langle e^{j\omega t}, f(t) \right\rangle$$

- Fourier synthesis formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Properties of Fourier Transform (1)

- Linearity

$$\alpha f(t) + \beta g(t) \leftrightarrow \alpha F(\omega) + \beta G(\omega)$$

- Shifting

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(\omega)$$

$$e^{j\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0)$$

- Scaling

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

- Differentiation / Integration

$$\frac{\partial^n f(t)}{\partial t^n} \leftrightarrow (j\omega)^n F(\omega) \qquad \int_{-\infty}^{t} f(\tau) d\tau \leftrightarrow \frac{F(\omega)}{j\omega}$$

Properties of Fourier Transform (2)

- Convolution of two functions

$$h(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = f(t) * g(t)$$

- *Convolution theorem*

$$f(t) * g(t) \leftrightarrow F(\omega) G(\omega)$$

- Complex exponentials are eigenfunctions of the convolution operator

$$\int_{-\infty}^{\infty} e^{j\omega(t-\tau)} g(\tau) d\tau = e^{j\omega t} \int_{-\infty}^{\infty} e^{-j\omega\tau} g(\tau) d\tau = e^{j\omega t} G(\omega)$$

Properties of Fourier Transform (3)

- Because the Fourier Transform is an orthogonal transform, it satisfies an energy conservation relation known as *Parseval's Formula*

$$\int_{-\infty}^{\infty} f^*(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)G(\omega)d\omega$$

when g(t) = f(t),

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

Fourier Series

- Given a periodic function f(t) with period T,

$$f(t+T) = f(t)$$

- Synthesis Formula

$$f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_0 t}, \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

- Analysis Formula

$$F[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jk\omega_0 t} \, dt$$

Dirac Function

- Defined as a limit of rectangular functions
- Infinitesimally narrow, infinitely tall, yet it integrates to unity
- Some relations

$$\Rightarrow \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f(t-t_0)\delta(t)dt = \int_{-\infty}^{\infty} f(t)\delta(t+t_0)dt = f(t_0)$$

$$\Rightarrow f(t) * \delta(t-t_0) = f(t-t_0)$$

$$\Rightarrow \delta(t-t_0) \leftrightarrow e^{-j\omega t_0}$$

Impulse Train

- Train of Dirac functions spaced T $>$ 0 apart, given by

$$s_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Fourier Transform of impulse train

$$S_T(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$$

Sampling

- Central to discrete-time signal processing, since it provides link to continous-time domain
- Call $f_T(t)$ the sampled version of $f(t)$, obtained as,

$$f_T(t) = f(t)\, s_T(t) = \sum_{n=-\infty}^{\infty} f(nT)\, \delta(t - nT)$$

- And the Fourier Transform of the sampled time signal is,

$$F_T(\omega) = F(\omega) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(\omega - k\frac{2\pi}{T}\right)$$

Sampling Theorem

- If $f(t)$ is continous and bandlimited to ω_m, then $f(t)$ is uniquely defined by its samples taken at twice ω_m. The minimum sampling frequency is $\omega_s = 2\omega_m$.

  $f(t)$ can be recovered by the following interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}_T(t-nT)$$

Alternative view on sampling

- Sinc-functions form an orthonormal system
- Standard sampling system (including anti-aliasing prefilter) may be interpreted as an orthogonal projection of not-necessarily band-limited input signals onto the space of band-limited signals
- Different interpretation of reconstruction formula

$$f(t) = \sum_{n \in Z} f(nT) \operatorname{sinc}(t/T - n)$$

- Due to the orthogonality the projection into the space of bandlimited signals yields the minimum-error approximation

Discrete Fourier Transform (1)

- Very important for computational reasons - can be implemented using the *FFT*
- The DFT consists of inner products of the input signal f with sampled complex sinusoids
- Analysis formula

$$F[k] = \sum_{n=0}^{N-1} f[n]\, e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} f[n]\, W_N^{nk}$$

- Synthesis formula

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k]\, W_N^{-nk}$$

Discrete Fourier Transform (2)

- Can be thought of as the transform of one period of a periodic signal, or a sampling of the DTFT of a finite signal
- The DTFT is a function of *continous* frequency whereas the DFT is a function of *discrete* frequency
- DFT can also be formulated as a complex matrix multiply

$$X(\omega_k) \triangleq \langle \underline{x}, \underline{s}_k \rangle \triangleq \sum_{n=0}^{N-1} \underline{x}(n) e^{-j2\pi nk/N}, \quad k = 0, 1, 2, \ldots, N-1$$

$$\underbrace{\begin{bmatrix} X(\omega_0) \\ X(\omega_1) \\ X(\omega_2) \\ \vdots \\ X(\omega_{N-1}) \end{bmatrix}}_{\underline{X}} = \underbrace{\begin{bmatrix} \overline{s_0(0)} & \overline{s_0(1)} & \cdots & \overline{s_0(N-1)} \\ \overline{s_1(0)} & \overline{s_1(1)} & \cdots & \overline{s_1(N-1)} \\ \overline{s_2(0)} & \overline{s_2(1)} & \cdots & \overline{s_2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{s_{N-1}(0)} & \overline{s_{N-1}(1)} & \cdots & \overline{s_{N-1}(N-1)} \end{bmatrix}}_{\mathbf{S}_N^*} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\underline{x}}$$

Frequency, Scale and Resolution (1)

- Fourier transform and it's variations are very useful tools, but practical applications require basic modifications
- "Localization" of the analysis is needed
  - not necessary to have the signal over $(-\infty, \infty)$ to perform the transform
  - local effects (transients) can be captured with some accuracy
- Important concept in this context is the *uncertainty principle*

Frequency, Scale and Resolution (2)

- Various ways to define the localization of a particular basis function, but they are all related to the "‘spread’" of the function in time and frequency

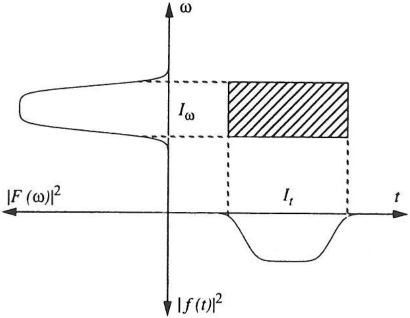


Figure 2.9 Tile in the time-frequency plane as an approximation of the time-frequency localization of $f(t)$. Intervals I_t and I_ω contain 90% of the energy of the time- and frequency-domain functions, respectively.

Frequency, Scale and Resolution (3)

- Basic operations: shifting, modulation, scaling

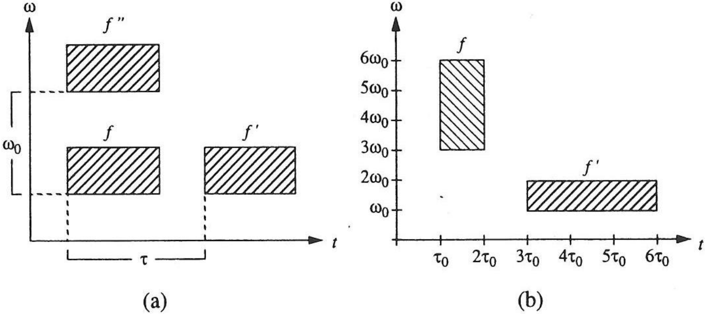


Figure 2.10 Elementary operations on a basis function f and effect on the time-frequency tile. (a) Shift in time by τ producing f' and modulation by ω_0 producing f''. (b) Scaling $f'(t) = f(at)$ ($a = 1/3$ is shown).

Uncertainty Principle

- Sharpness (resolution) of time analysis can be traded off for sharpness in frequency, and vice versa
- Measure for width in time and frequency

$$\Delta_t^2 = \int_{-\infty}^{\infty} t^2 \left|f(t)\right|^2 dt$$

$$\Delta_\omega^2 = \int_{-\infty}^{\infty} \omega^2 \left|F(\omega)\right|^2 dt$$

- If the waveform is well-localized in both frequency and time, then the *time-bandwidth product* $\Delta_t \Delta_\omega$ will be small
- *Uncertainty Principle*
  If f(t) is differentiable and vanishes faster than $\frac{1}{\sqrt{|t|}}$ as $t \to \pm\infty$,

$$\Delta_t \Delta_\omega \geq \frac{1}{2}$$

Short-Time Fourier Transform and Wavelet Transform (1)

STFT

- "local" Fourier transform

$$STFT_f(\omega,\tau) = \int_{-\infty}^{\infty} w^*(t-\tau) f(t) e^{-j\omega t} dt$$

$$STFT_f(\omega,\tau) = \langle g_{\omega,\tau}(t), f(t) \rangle$$

$$\text{where } g_{\omega,\tau}(t) = w(t-\tau)e^{j\omega t}$$

Wavelet Transform

- Basis function usually is a bandpass filter that is shifted and scaled

$$CWT_f(a,b) = \frac{1}{\sqrt{a}} \int_R \psi^*\left(\frac{t-b}{a}\right) f(t) dt$$

$$CWT_f(a,b) = \langle \psi_{a,b}(t), f(t) \rangle$$

Short-Time Fourier Transform and Wavelet Transform (2)

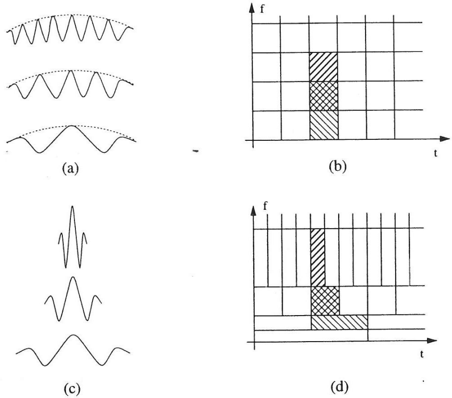


Figure 2.12 The short-time Fourier and wavelet transforms. (a) Modulates and shifts of a Gaussian window used in the expansion. (b) Tiling of the time-frequency plane. (c) Shifts and scales of the prototype bandpass wavelet. (d) Tiling of the time-frequency plane.

Conclusion

- Inner product is used to project one vector onto another
- Every signal expansion can be seen as a projection onto a Hilbert Space
- Problem with Fourier transform - no localization in time
- Modifications aim at ”localizing” the analysis
- Not possible to become a arbitrarly sharp resolution in both domains simultaneously