Fundamentals of Signal Decompositions

Andreas Lessiak

Graz University of Technology

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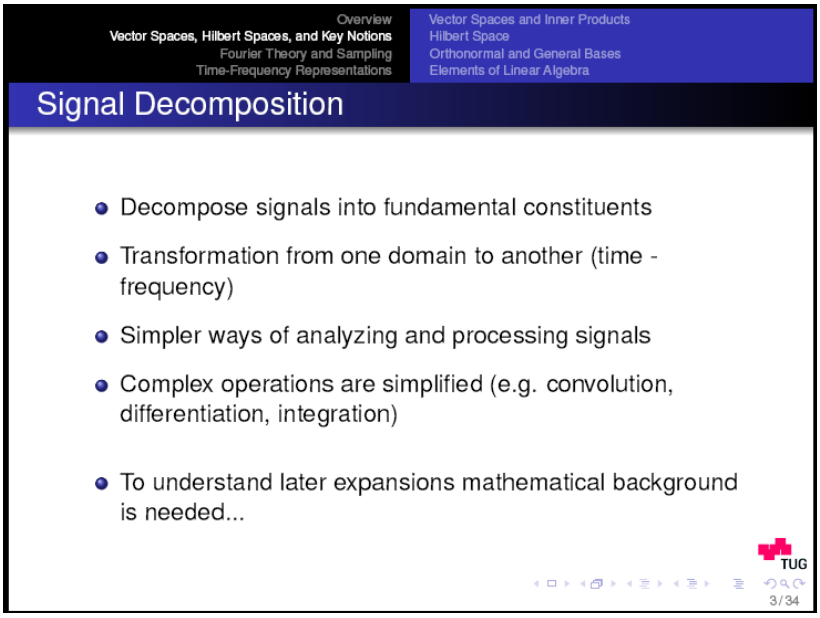
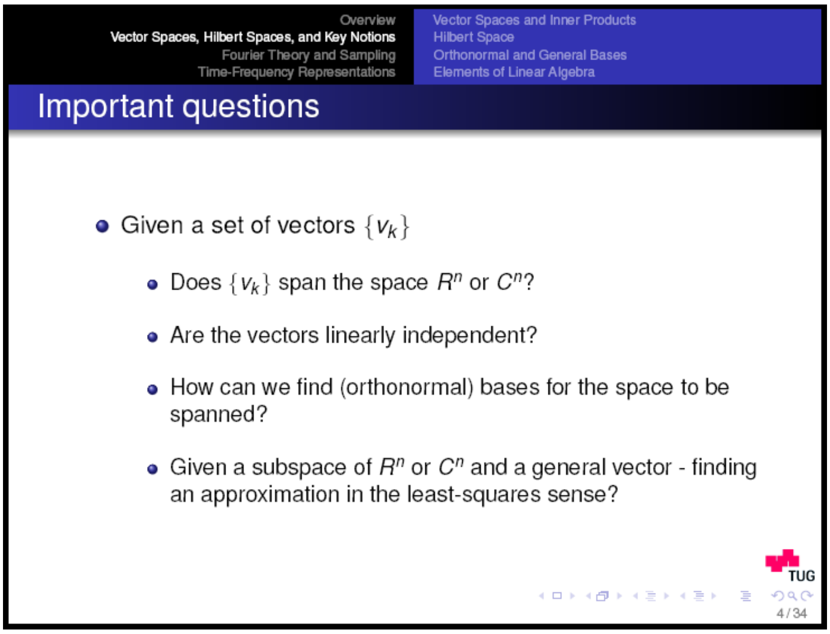
Overview

- Vector Spaces, Hilbert Spaces, and Key Notions
- Basic concepts from Linear Algebra
- Fourier Theory and Sampling
- Time-Frequency Representations



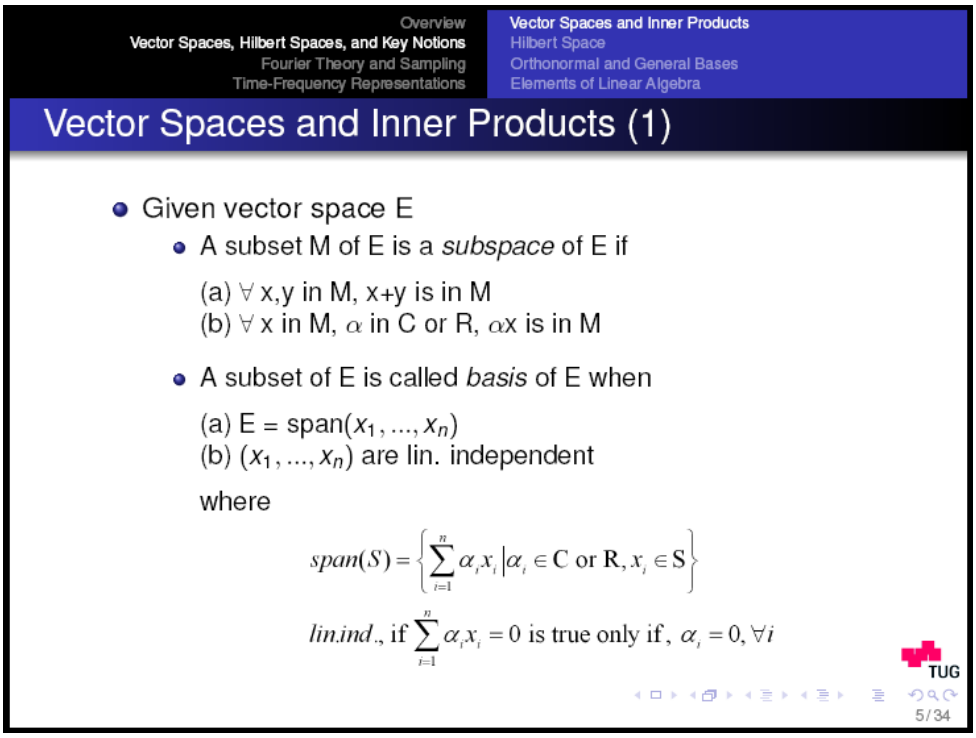
A common theme in many signal processing applications is to decompose a signal into its primitive or fundamental constituents and perform simple operations separately on each component. Signal decomposition or expansion, as it is often called, can also be seen as a transformation of the original signal from one domain to another (e.g. time to frequency). This transformation then helps in analyzing and processing the signal, e.g. it is possible to analyze which frequency region contains most of the energy. Also many complicated operations have simpler equivalents in the other domain (e.g. differentiation in case of the Fourier transform becomes a multiplication). But to understand the notions used for the expansions that are explained later in this document one also needs to have the mathematical framework.



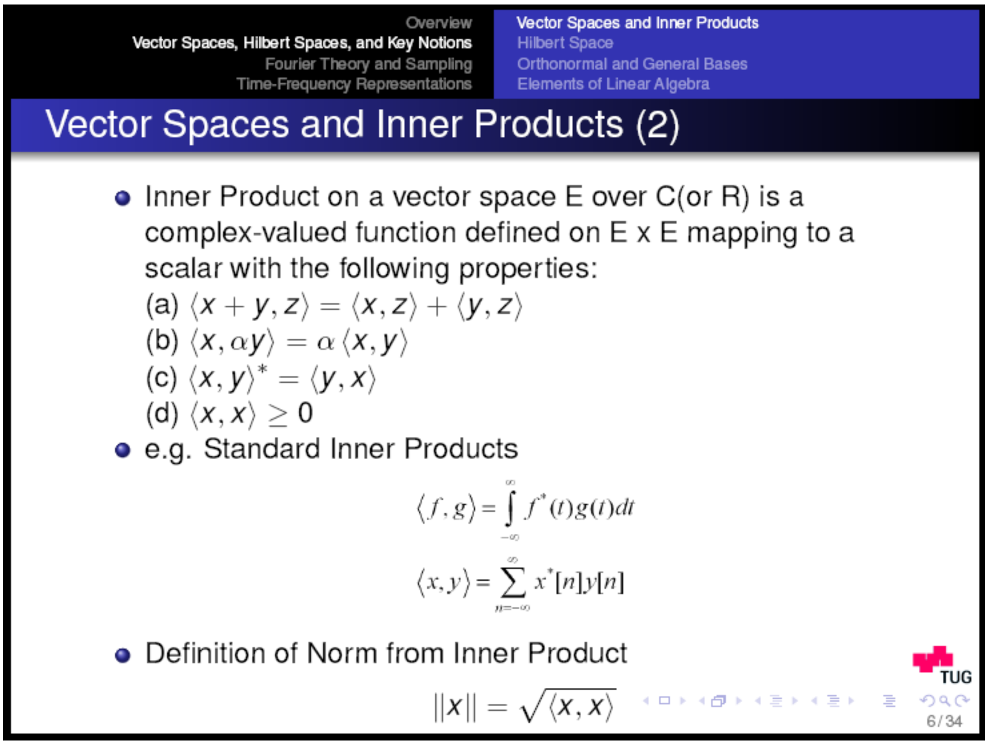
When dealing with these questions two key notions to address them are

- The length, or norm, of a vector
- The orthogonality of one vector with respect to another

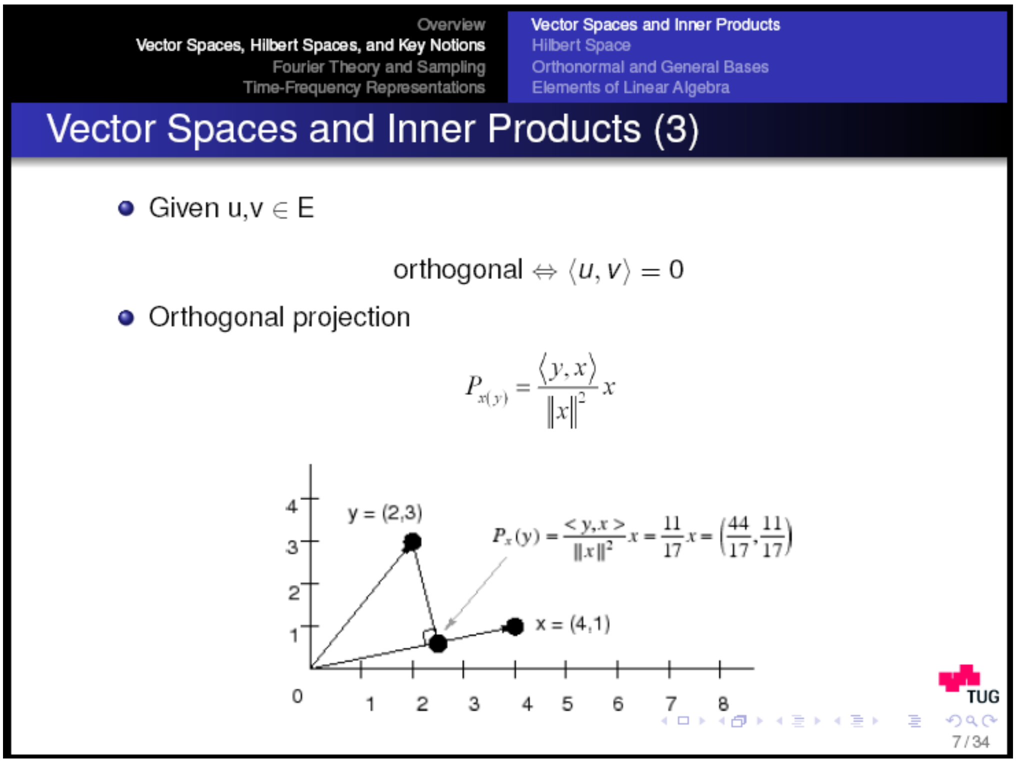
The ideas behind them can then be generalized to infinite spaces where we restrict the vectors to have finite length or norm (even though they are infinite-dimensional)



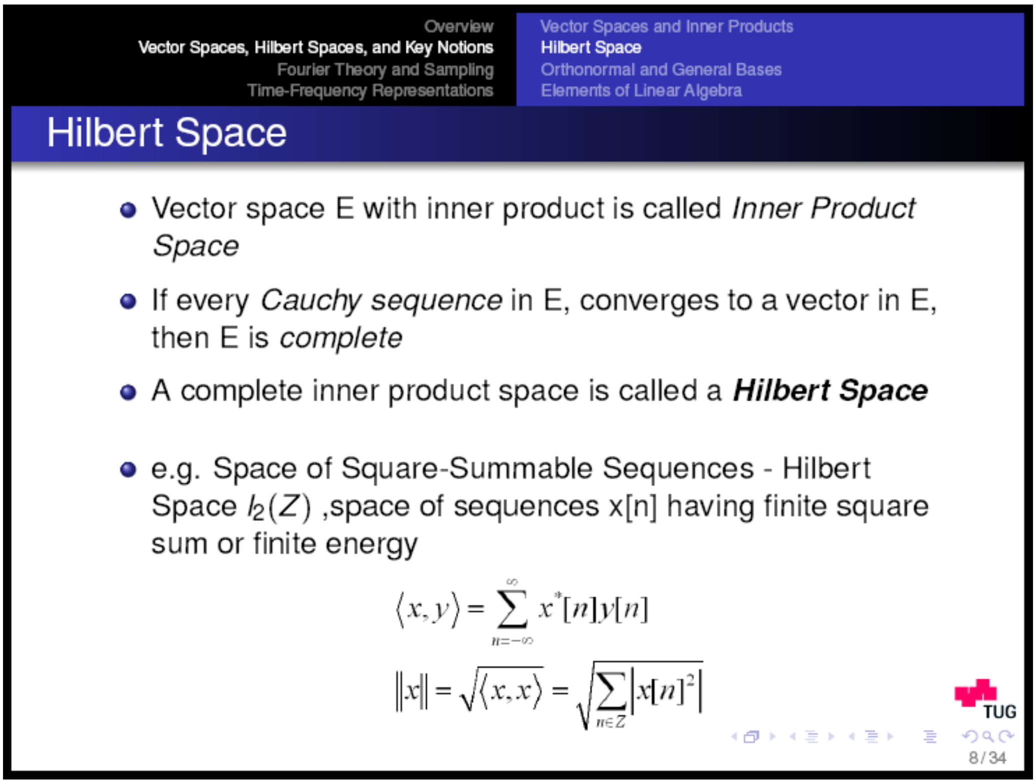
Important and often encountered notions when talking about vector spaces are the notion of a *subspace* and a *basis* of a vector space. For the definition of a basis one needs to know what is meant by the *span(S)* of a set of vectors and by *linear independence* of a set of vectors. The span of a set of vectors is the set of vectors that can be given as a linear combination of vectors in S. Linear independence of a vector set S signifies that no vector in S can be build by a linear combination of the other vectors in S.



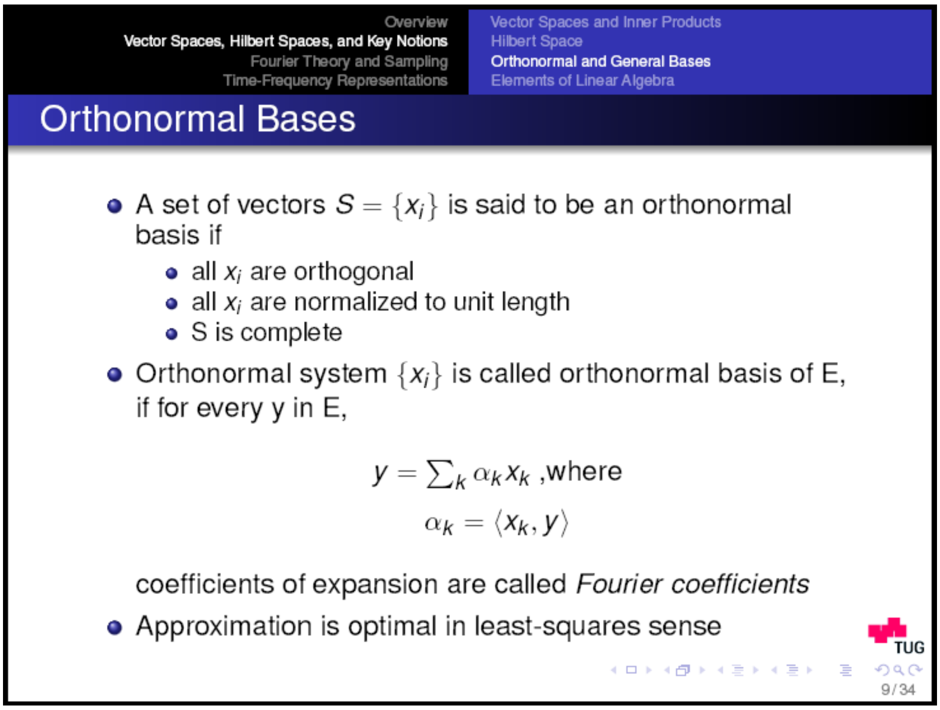
In simple terms, the inner product measures the relative alignment (angle in case of Euclidean space) between two vectors. This adds additional structure to a given vector space. The inner product defines a norm, although not every norm can be defined by an inner product.



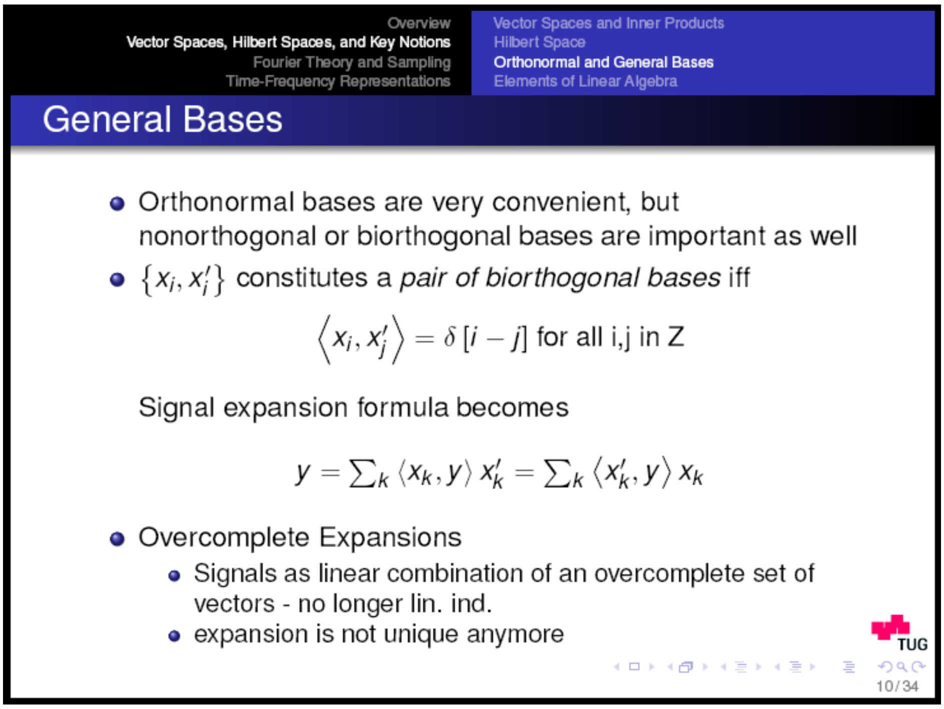
Inner product is also used to define *orthogonality* and for projecting one vector onto another vector. A set of vectors is called orthogonal if the vectors are pair wise orthogonal. If they are normalized to unit norm the vectors form an *orthonormal* system.



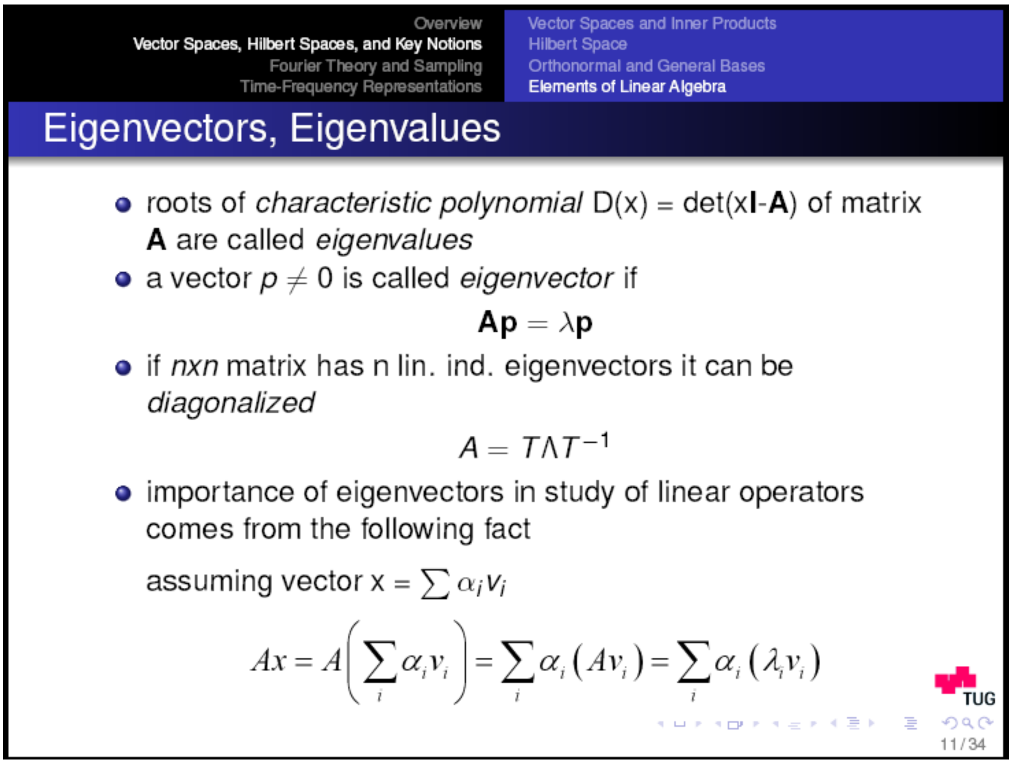
The question of completeness of a space deals with the question, whether the span of a set of vectors covers the whole space or not. In other words this means that every vector in the space can be represented as a linear combination of the basis vectors. A Cauchy sequence of vectors is a sequence where the distance between its elements eventually becomes arbitrarily small (the sequence converges).



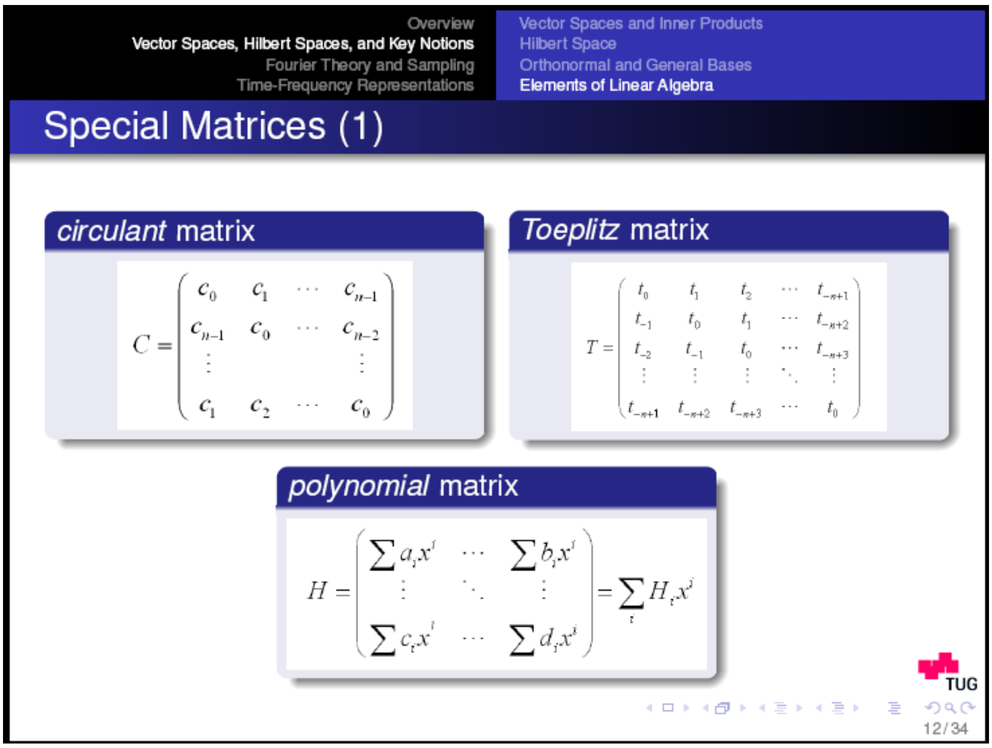
Among all possible bases in a Hilbert space, orthonormal bases play a very important role. Every arbitrary basis can be orthogonalized by a procedure from linear algebra, called Gram-Schmidt Orthogonalization. If we have an orthonormal basis to a vector space E, then every vector can be represented by a linear combination of the basis vectors, where the coefficients are the projection coefficients of the vector onto each basis vector and are called Fourier coefficients. If we are trying to approximate a vector from a higher-dimensional Hilbert space E by a vector lying in a subspace S of E, the orthogonal projection onto S yields an approximation that is optimal in the least-squares sense.



The term biorthogonal is used since to the (nonorthogonal) basis corresponds a dual basis which satisfies the biorthogonality constraint. If the basis is orthogonal, then it is its own dual. Overcomplete expansions are the starting point when talking about *frames*.



Every Matrix can be seen as a *linear operator* representing a system. If a vector, which is fed to the system, is only scaled, then this vector is called an *eigenvector* of the system. The corresponding scaling factor is called an *eigenvalue* of the system. The concept of eigenvectors generalizes to *eigenfunctions* for continous operators. A classic example is the complex sinusoid.



Many matrices exist that have a special structure. Polynomial matrices are often used in Signal Processing, because a FIR filter system can be represented by a polynomial. Also IIR filter systems can be represented by matrices of rationals.

Special Matrices (2)

- An example for a matrix with toeplitz structure is the autocorrelation matrix

$$R_{xx} = \begin{pmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & \cdots \\ r_{xx}[-1] & r_{xx}[0] & r_{xx}[1] & \cdots \\ r_{xx}[-2] & r_{xx}[0] & r_{xx}[0] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $r_{xx}[m] = E\{x[n]\,x[n+m]\}$

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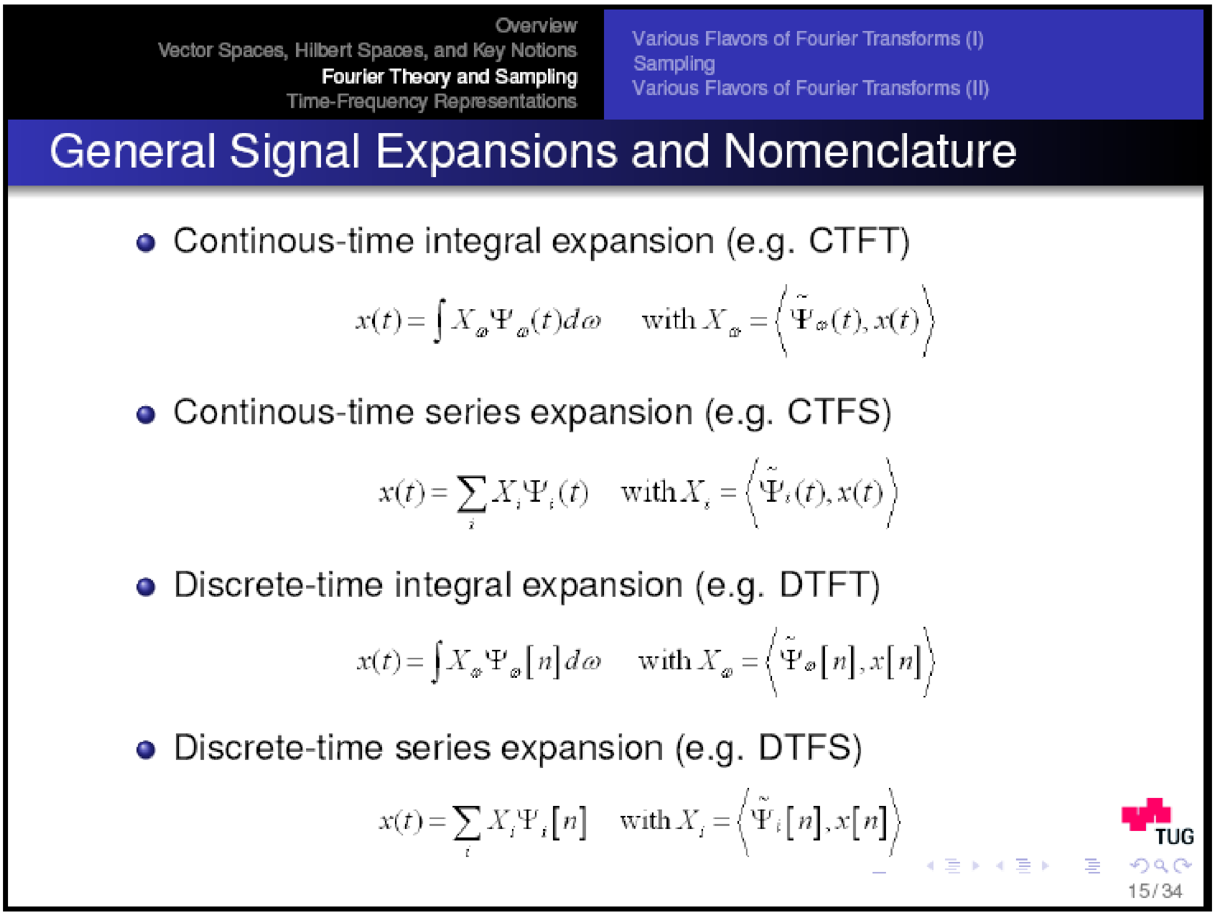
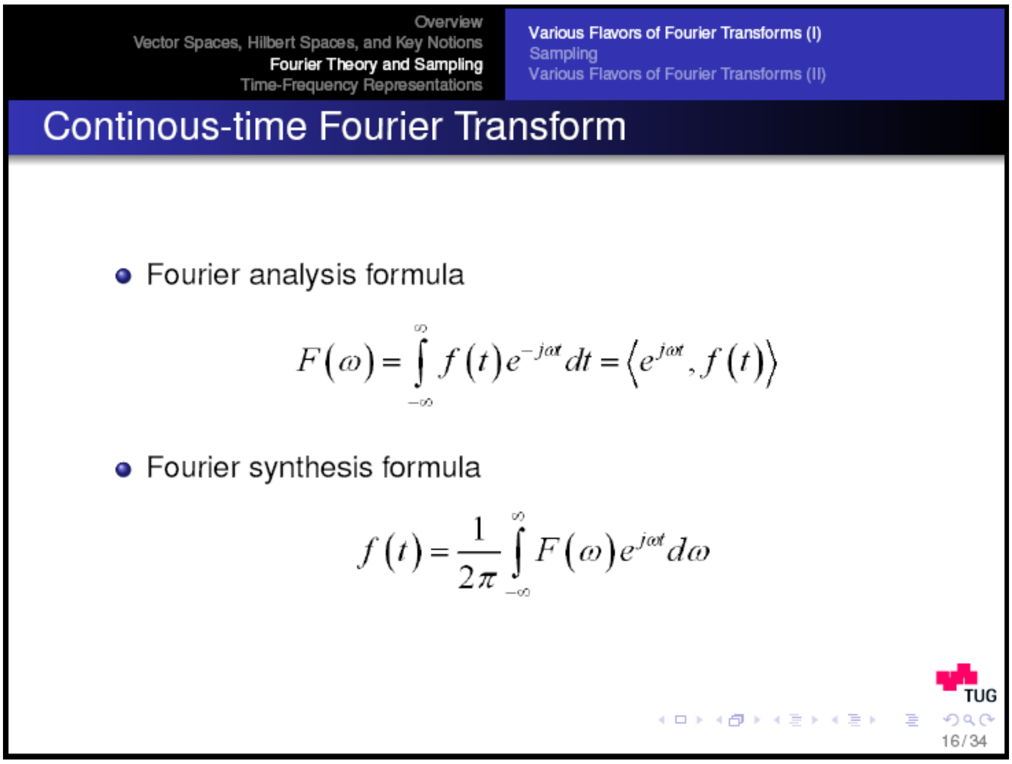
Vector Spaces and Inner Products
Hilbert Space
Orthonormal and General Bases
Elements of Linear Algebra

What we've done so far...

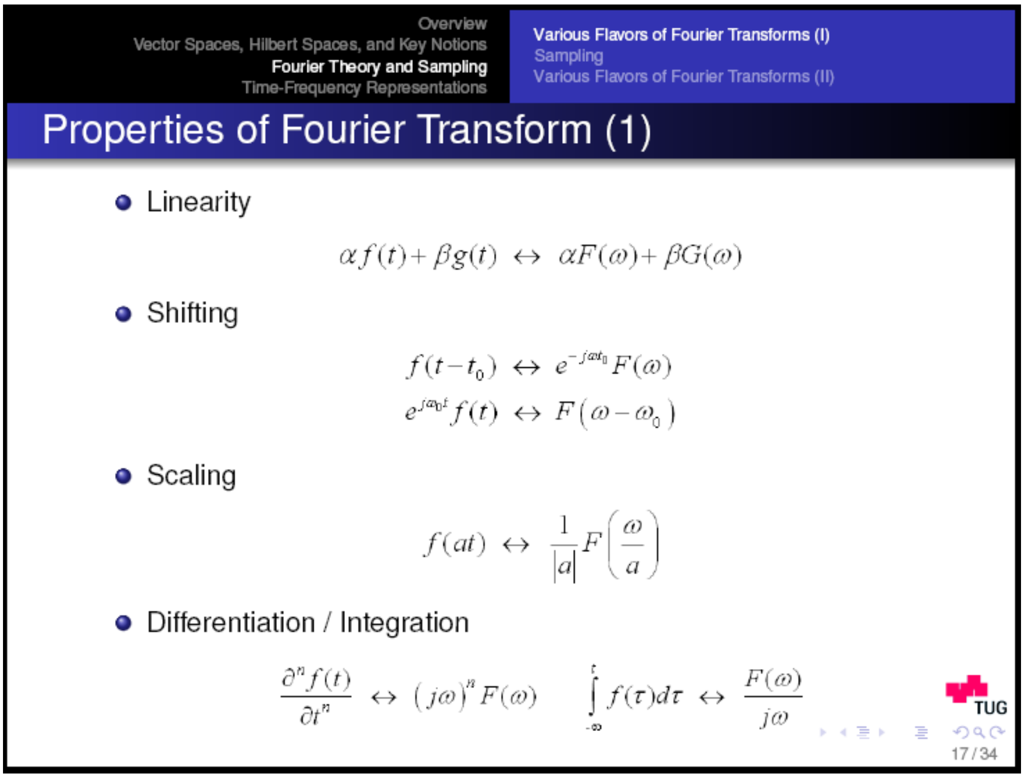
- ... Inner Product
- ... Hilbert Space
- ... Projection
- ... Orthonormal and general bases
- ... Eigenvectors and Eigenvalues



As mentioned before the inner product is used to project one vector onto another. Each expansion is just the projection (using an inner product) of the input onto a corresponding basis. Depending on the type of basis functions and the input, e.g. whether they are continuous or discrete, different types of signal expansions are used. Be aware of the usage of biorthogonal bases in the above equations.



Given an absolutely integrable function f(t), the Fourier analysis and synthesis formulas are defined like above. The factor 1/(2π) in the synthesis formula comes from the usage of ω and can be easily explained when using the frequency f and making a variable substitution. The continuous-time Fourier transform is a projection, using an inner product, onto complex sinusoids used as a basis.



The Fourier transform satisfies a number of properties, some of which are shown above.

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Various Flavors of Fourier Transforms (I)
Sampling
Various Flavors of Fourier Transforms (II)

Properties of Fourier Transform (2)

- Convolution of two functions

$$h(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = f(t) * g(t)$$

- *Convolution theorem*

$$f(t) * g(t) \leftrightarrow F(\omega) G(\omega)$$

- Complex exponentials are eigenfunctions of the convolution operator

$$\int_{-\infty}^{\infty} e^{j\omega(t-\tau)} g(\tau) d\tau = e^{j\omega t} \int_{-\infty}^{\infty} e^{-j\omega\tau} g(\tau) d\tau = e^{j\omega t} G(\omega)$$

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As mentioned earlier some operations in one domain have simpler equivalents in the other domain.

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Various Flavors of Fourier Transforms (I)
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Various Flavors of Fourier Transforms (II)

Properties of Fourier Transform (3)

- Because the Fourier Transform is an orthogonal transform, it satisfies an energy conservation relation known as *Parseval's Formula*

$$\int_{-\infty}^{\infty} f^*(t) g(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) G(\omega) d\omega$$

when g(t) = f(t),

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

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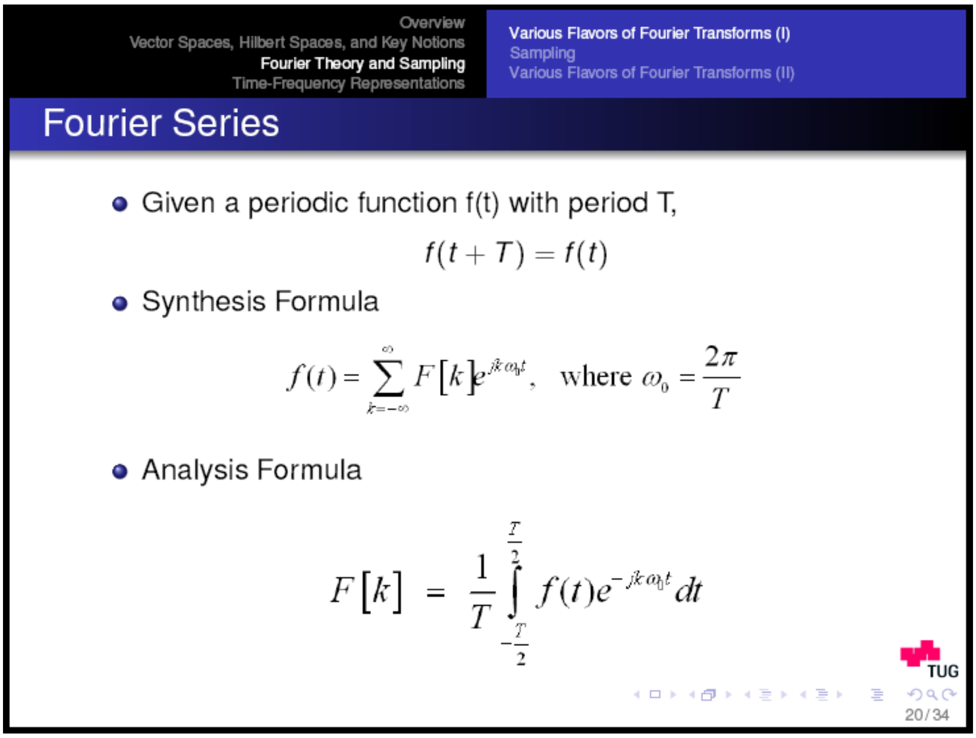
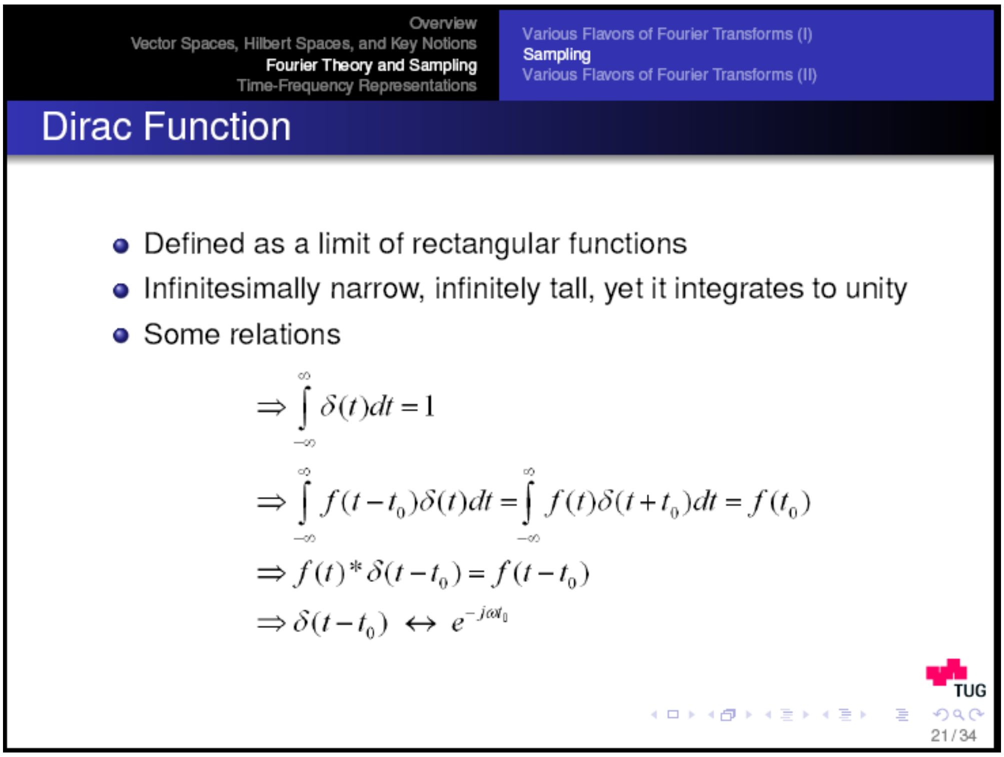
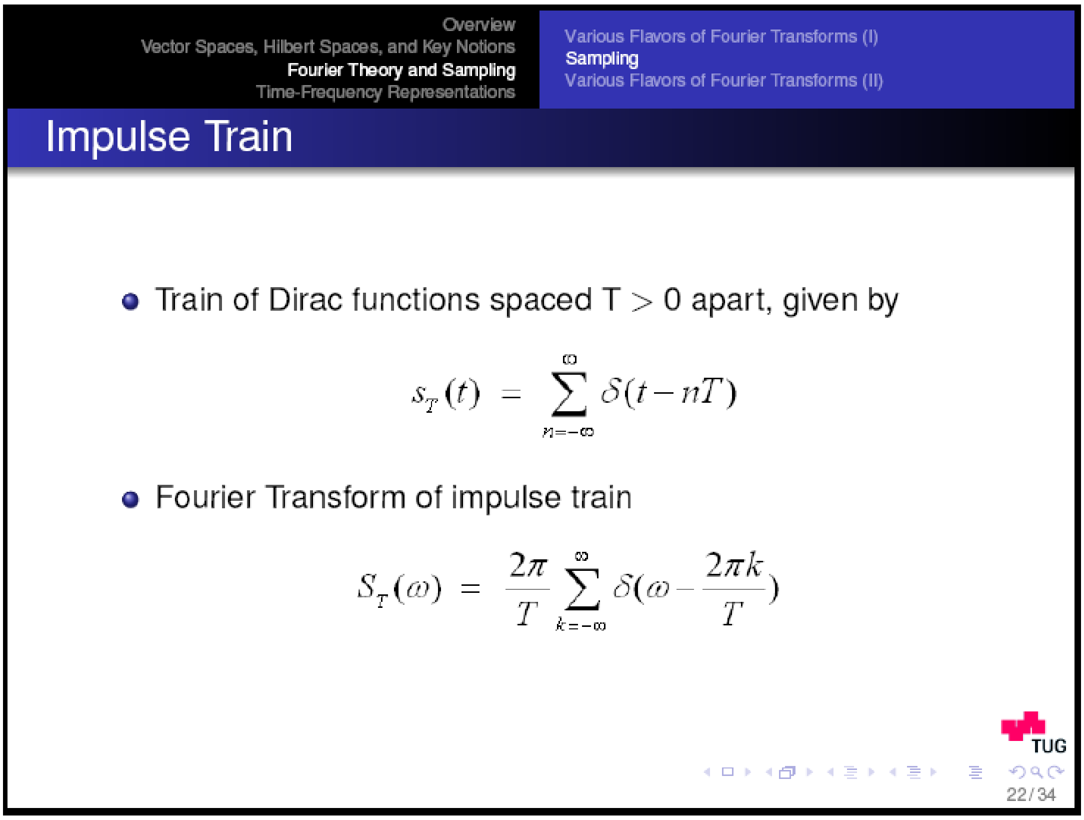


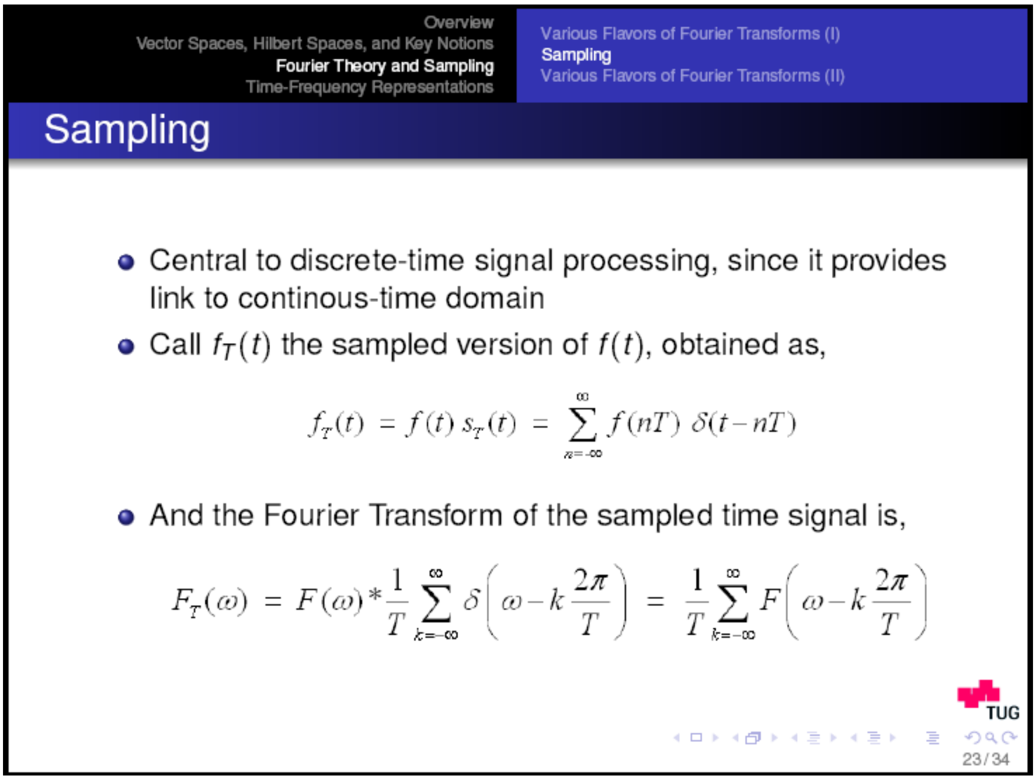
Fourier series, beside their obvious use for characterizing periodic signals, are useful for problems of finite size through periodizitation. A problem which then occurs is the introduction of discontinuities at the boundary, since in general, periodizitation of a continuous signal on an interval results in a discontinuous periodic signal.



More generally, any smooth function can be used to define the Dirac delta. While the Dirac delta function in the continuous domain has to be treated with care, its equivalent, the discrete Dirac delta impulse is simply defined to be a one at time instant zero, and zero otherwise, in the discrete domain.

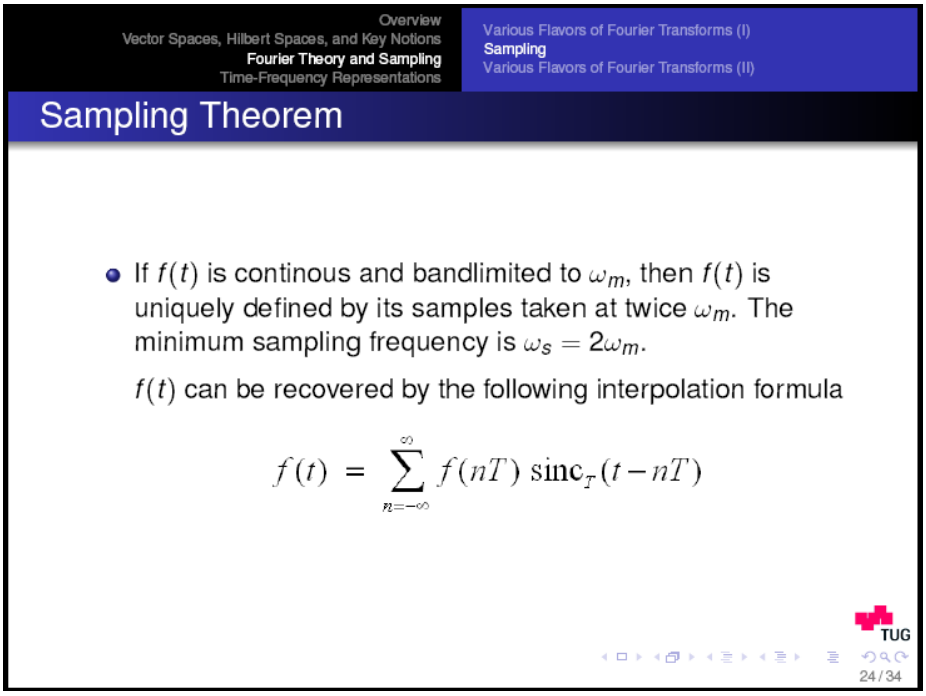


The impulse train is a very important tool in sampling theory.

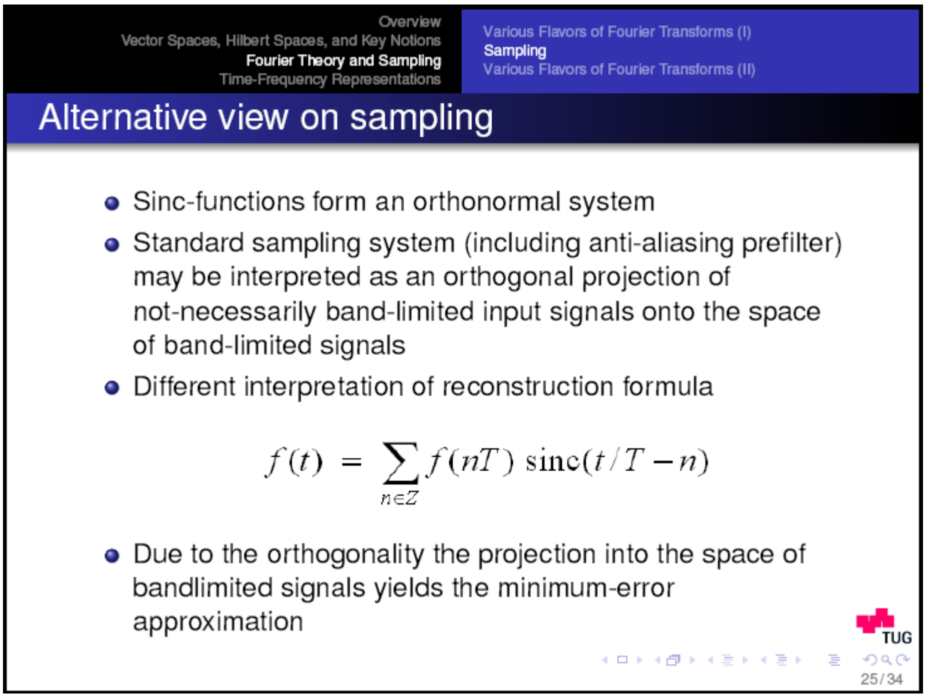


Sampling the spectrum and periodizing the time-domain function are equivalent. The same holds for the dual situation, where sampling the time-domain function leads to a periodized spectrum.
A sampled signal can also be seen as a continuous signal multiplied by the above defined impulse train. In the frequency domain this multiplication results in a convolution. Therefore

the spectrum of the sampled signal is the periodized version of the spectrum of the continuous signal. The copies of the spectrum are spaced $\frac{2\pi}{T}$ apart.



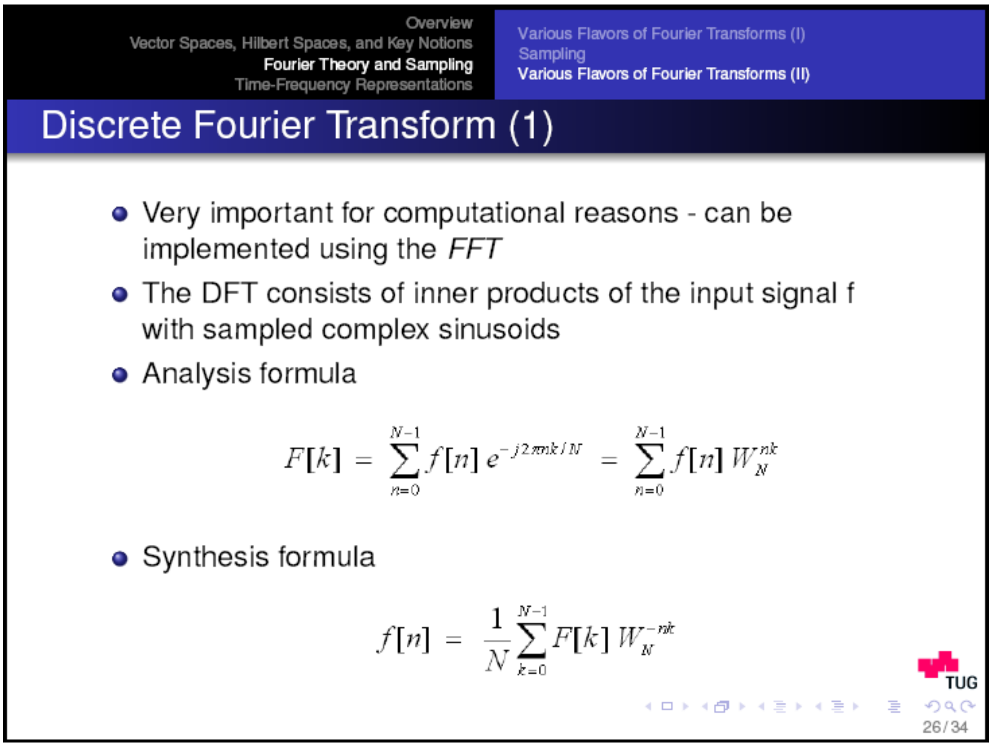
The famous theorem by Whitaker, Kotelnikov and Shannon states that a bandlimited signal can be completely reconstructed if the sampling frequency is twice the maximal signal frequency. Note that the sinc – function has the interpolation property since it is 1 at the origin but 0 at nonzero multiples of T.



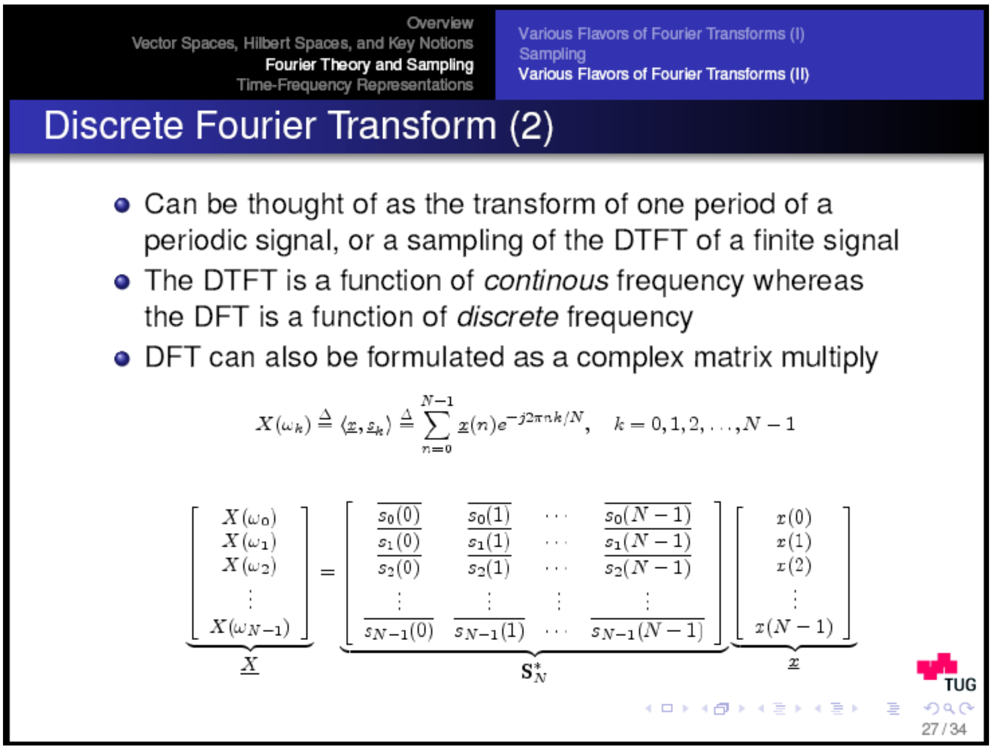
An alternative interpretation of the sampling theorem is as a series expansion on an orthonormal basis for bandlimited signals. Another way of writing the interpolation formula is

$$f(t) = \sum_{n=-\infty}^{\infty} \left\langle \varphi_{n,T}, f \right\rangle \varphi_{n,T}$$

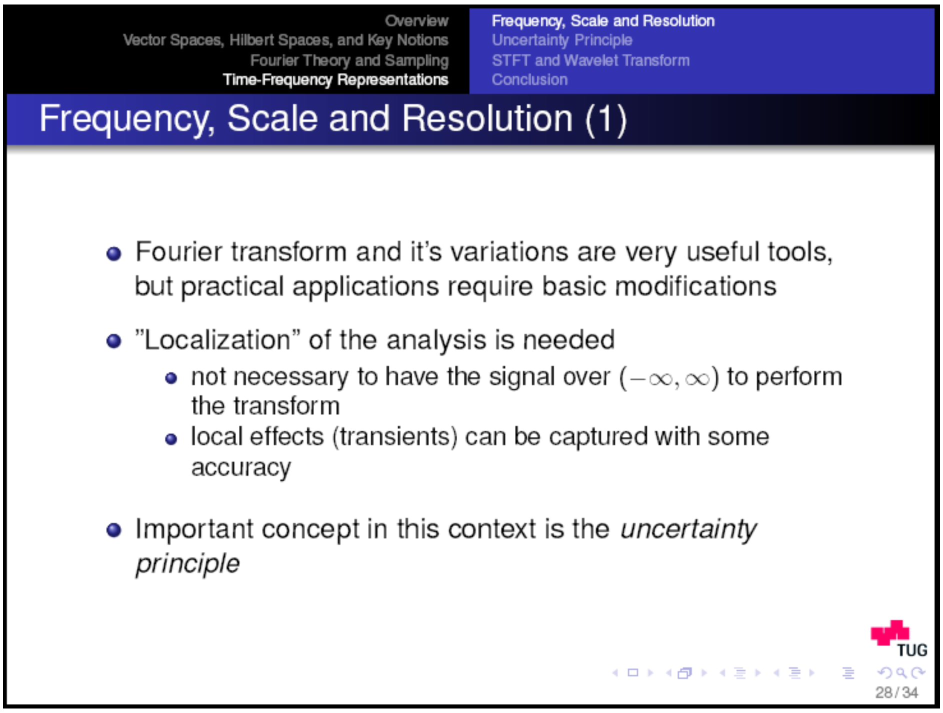
where, $\varphi_{n,T} = \frac{1}{\sqrt{T}} \mathrm{sinc}_T(t - nT)$ form an orthonormal basis for the space of bandlimited functions.



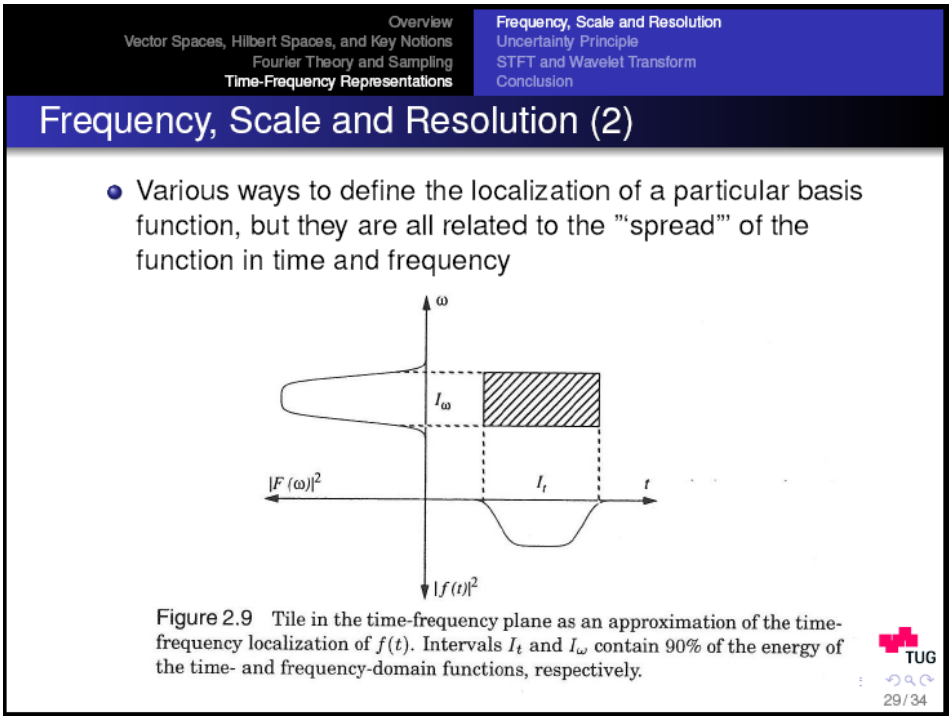
There is a close relationship between the DTFT and the DFT. The DFT can be thought of as the transform of one period of a periodic signal, or a sampling of the DTFT of a finite-length signal. In contrary to a DTFT f[n] and F[k] are not defined for $n, k \notin \{0, \ldots, N-1\}$ in the case of a DFT.



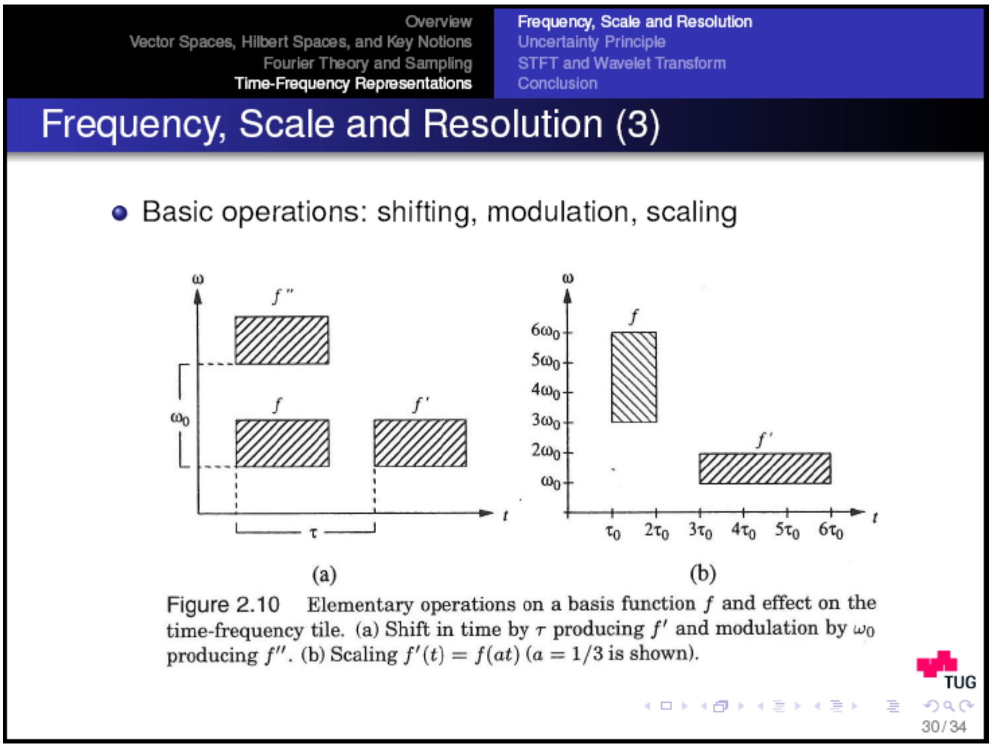
Because of the finite-length signals involved the DFT can be represented as a matrix-vector product.



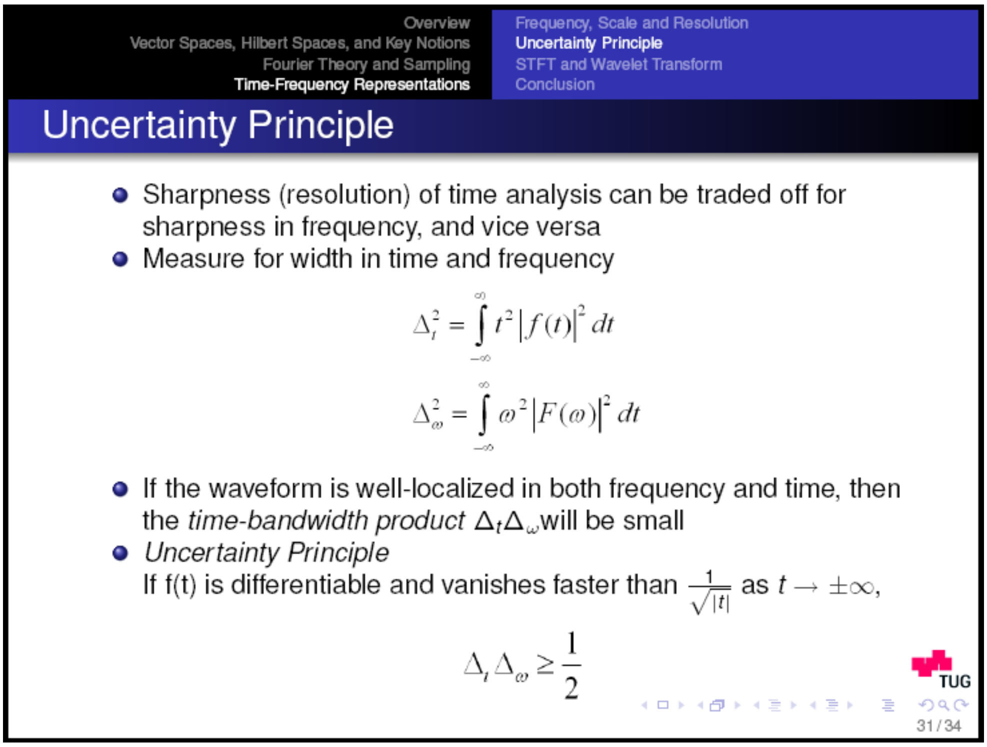
Fourier analysis is not always the best tool to analyze a signal whose characteristics vary with time. If, for example, a signal composed of two sinusoids with different frequencies and a glitch at time instant t_0 is fourier transformed the glitch causes a wide spread of frequency components and is therefore not efficiently described. Another simple example would be a time-domain signal consisting of one period of a sine – the sine is rectangulary windowed. Fourier theory says that a rectangular window results in a sinc – like frequency spreading. So more localized transforms are needed.



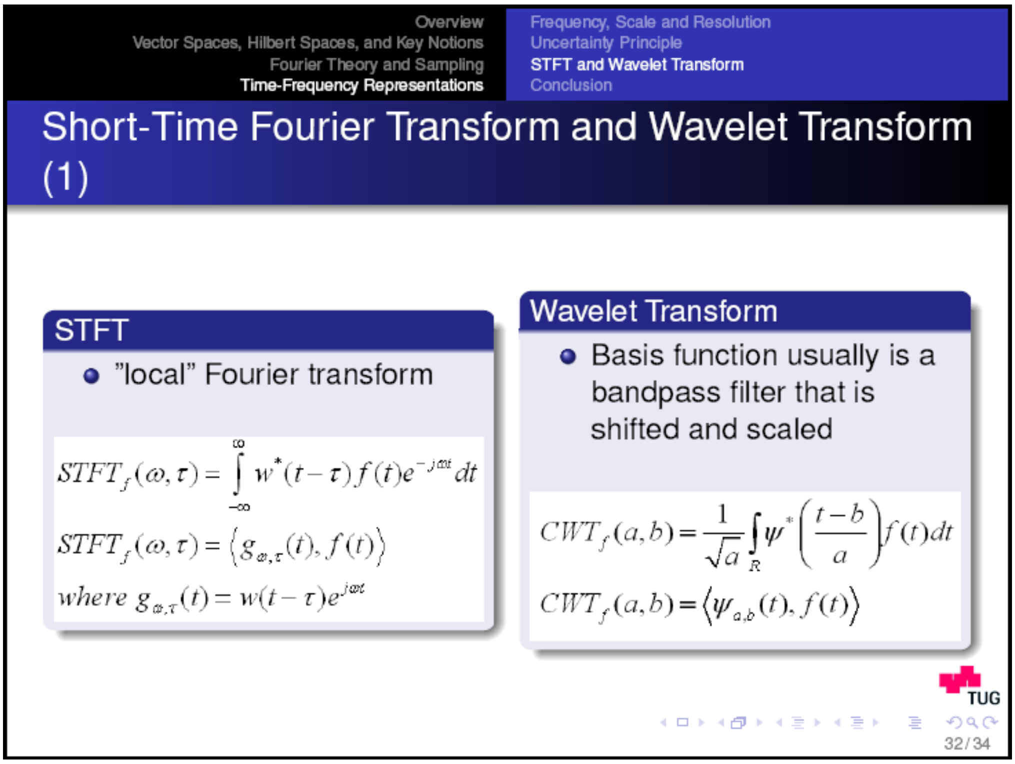
To define localization of a particular basis one can define intervals I_t and I_ω which contain 90% of the energy of the time- and frequency-domain functions, respectively, and are centered around the center of gravity of $|f(t)|^2$ and $|F(\omega)|^2$. This defines what is called a *tile* in the time-frequency domain.



Modulation by $e^{j\omega_0 t}$ or a shift by τ simply result in a translation of the tile in the corresponding direction. Scaling by a, or $f'(t) = f(at)$, results in $I'(t) = \frac{1}{a} I(t)$ and $I'(\omega) = aI(\omega)$. Therefore scaling affects both the shape and the localization. In the case of scaling as shown in the slide above resolution in time was traded for resolution in frequency.

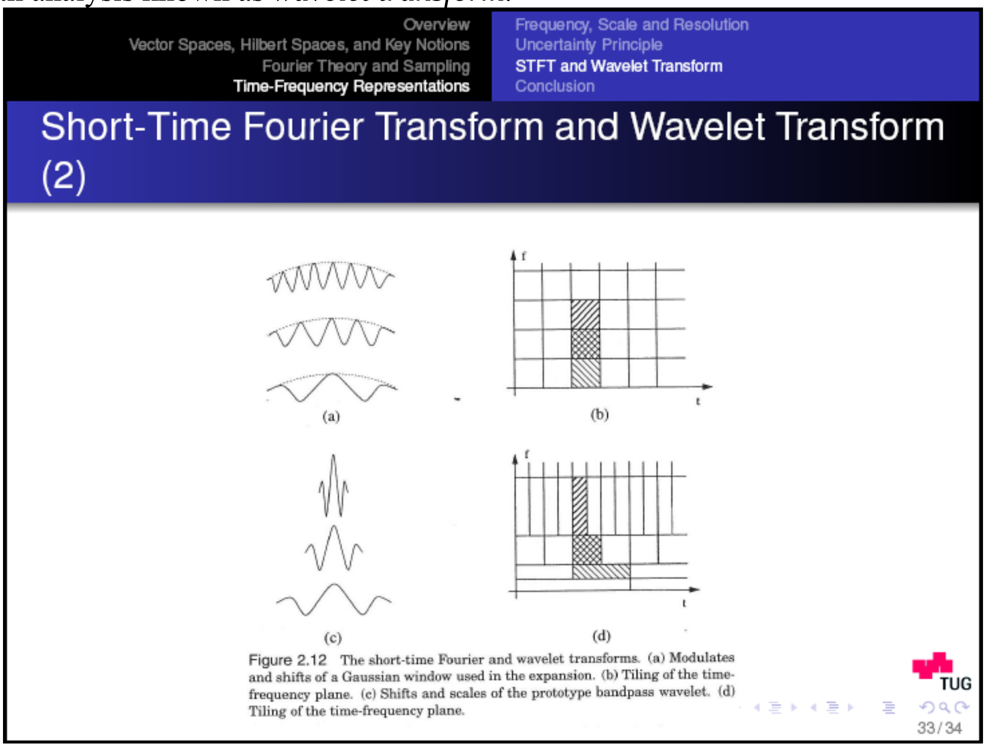


There is no way to get an arbitrarily sharp resolution in both domains simultaneously and the uncertainty principle gives a lower bound that can not be passed. If a waveform is well-localized in time it will have a small Δ_t and if it is well-localized in frequency it will have a small Δ_ω .



To achieve a "local" Fourier transform, one can define a windowed Fourier transform. The signal is first multiplied by a window function w(t-τ) (e.g. Hamming, Hanning, Gaussian,…) and then the usual Fourier transform is taken. That is, one measures the similarity between the signal and shifts and modulates of an elementary window. The *spectrogram* is the energy distribution associated with the STFT.

Instead of shifts and modulates one can choose shifts and scales of a prototype function (e.g. bandpass filter with zero mean and certain impulse response) and obtain a constant relative bandwidth analysis known as *wavelet transform*.



Conclusion

- Inner product is used to project one vector onto another
- Every signal expansion can be seen as a projection onto a Hilbert Space
- Problem with Fourier transform - no localization in time
- Modifications aim at ”localizing” the analysis
- Not possible to become a arbitrarily sharp resolution in both domains simultaneously