Fundamentals of Multirate Systems

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Abstract

There has been significant advancement in the field of digital signal processing over the past several decades. Classical digital signal processing structures are the so-called single-rate systems because the sampling rates are the same at all points of the system. There are many applications where the signal of a given sampling rate needs to be converted into an equivalent signal with a different sampling rate. Such systems are called multirate systems. This paper presents the fundamentals of multirate building blocks and filter banks and describes some applications of multirate systems.

1 Introduction

During the last several years, the multirate processing of digital signals has attracted many researchers. The multirate approach increases the computation speed, decreases the overall filter order, reduces word-length effects, and decreases power consumption. Consequently, one of the main characteristics of multirate systems is their high computational efficiency. Multirate digital signal processing has different applications, such as efficient filtering, subband coding of speech, audio and video signals, analog/digital conversion, communications etc.

The two basic operations in multirate digital signal processing are decimation and interpolation. These operations can be performed by building blocks known as decimators and expanders.

An M-fold decimator (Figure 1) that takes an input $x(n)$ and produces the output sequence[1]

$$y_D(n) = x(Mn) \tag{1}$$

where M is an integer. Decimation results in aliasing unless $x(n)$ is bandlimited in a certain way. In general, however, it may not be possible to recover $x(n)$ from $y_D(n)$ because of loss of information.

An L-fold expander (Figure 2) takes an input x(n) and produces an output

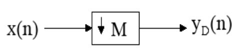

Figure 1: M-fold decimator

sequence [1]

$$y_E(n) = \begin{cases} x(n/L) & \text{if } n \text{ is integer multiple of } L \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Figure 2: L-fold expander

1.1 Transform Domain Analysis of Decimators and Expanders

1.1.1 Expander

We have [1]

$$\begin{aligned} Y_E(z) &= \sum_{n=-\infty}^{\infty} y_E(n) z^{-n} = \sum_{n=\text{mul. of } L} y_E(n) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} y_E(kL) z^{-kL} = \sum_{k=-\infty}^{\infty} x(k) z^{-kL} \\ &= X(z^L). \end{aligned} \tag{3}$$

So $Y(e^{j\omega}) = X(e^{j\omega L})$. This means that $Y(e^{j\omega})$ is an L-fold compressed version of $X(e^{j\omega L})$. Hence expander creats an imaging effect.

1.1.2 Decimator

For the M-fold decimator (1), we can write output $Y_D(e^{j\omega})$ in terms of $X(e^{j\omega})$ as [1]

$$Y_D(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M}) \tag{4}$$

This can be graphically interpreted as follows: (a) stretch $X(e^{j\omega})$ by a factor M to obtain $X(e^{j\omega/M})$, (b) create $M-1$ copies of this stretched version by shifting it uniformly in successive amount of 2π, and (c) add all these shifted stretched versions to the unshifted stretched version $X(e^{j\omega/M})$ and divide by M. The aliasing created by decimation can be avoided if $x(n)$ is a lowpass signal bandlimited to the region $|\omega| < \pi/M$.

1.2 Decimation and Interpolation Filters

In most applications, the decimator is preceded by a lowpass digital filter called the decimation filter. This filter ensures that the signal being decimated is bandlimited. The exact bandedges of the filter depend on how much aliasing is permitted. For example, in QMF banks a certain degree of aliasing is usually permitted because this can eventually be canceled off.

An interpolation filter is a digital filter that follows an expander.The typical purpose is to suppress all the images. Typically interpolation filter is lowpass

with cutoff frequency π/L.
Both types of filters with their typical responses are shown in Figure 3 and 4.

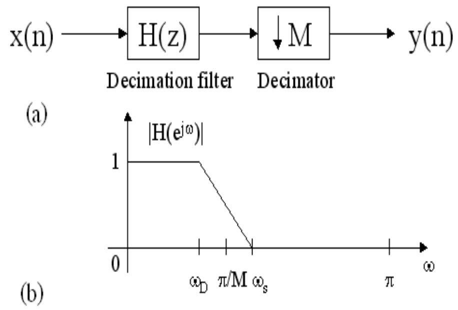

Figure 3: (a) The complete decimation circuit, and (b)typical response of the decimation filter

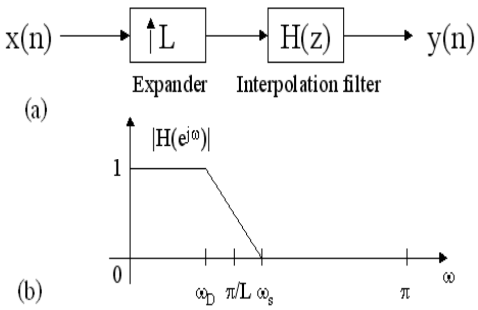

Figure 4: (a) The complete interpolation circuit, and (b)typical response of the interpolation filter

1.3 Fractional Sampling Rate Alteration

Uptill now we have seen that sampling rate of a signal can be altered by an integer factor. In certain applications, it is required to change the sampling rate by a rational fraction. An example of such a system is shown in Figure. 5. Here $L = 2, M = 3$, so $M/L = 1.5$. Here $X(e^{j\omega})$ is bandlimited to $|\omega| \leq 2\pi/3$. If signal is decimated by two, then that would create aliasing error. But it is possible to decimate by factor 1.5 (as shown in the Figure. 5 by broken lines)

1.4 Digital Filter banks

A digital filter bank is a collection of digital filters having common input or common output. There are two types of filter banks known as *analysis bank* and *sysntesis bank*. An analysis bank together with analysis filters $H_k(z)$ splits a signal $x(n)$ into M subband signals $x_k(n)$, while the task of a synthesis bank is to combine the M subband signals into a single signal $x^{'}(n)$ using the synthesis filters $F_k(z)$. Both types of filter banks have been shown in Figure. 6.

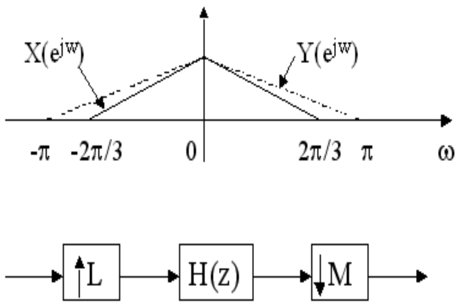

Figure 5: Fractional decimation

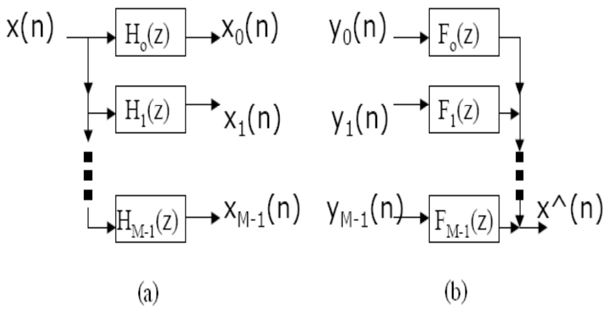

Figure 6: (a)Anaysis Bank (b) Synthesis Bank

2 Interconnection of Building Blocks

Figure. 7 shows the interconnection of commonly used building blocks in multirate systems.

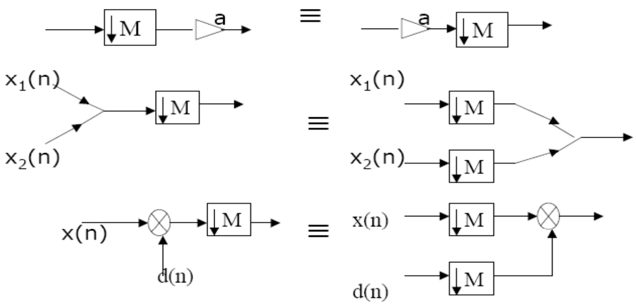

Figure 7: Simple Identities for interconnected systems

2.1 Noble Identities

Noble Identities are very useful in the theory and implementation of multirate systems(c.f Figure. 8).

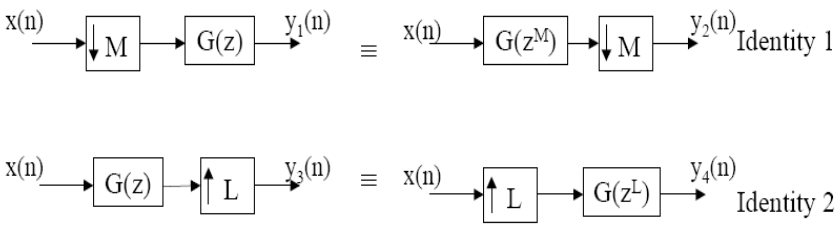

Figure 8: Noble Identities for multirate systems

3 The Polyphase Representation

The polyphase representation simplifies the theoratical results and leads to more efficient implementations of decimation/interpolation filters as well as filter banks. Consider a filter

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

By separating the even numbered coefficients of $h(n)$ from odd numbered ones, we get [1]

$$H(z) = \sum_{n=-\infty}^{\infty} h(2n)z^{-2n} + \sum_{n=-\infty}^{\infty} h(2n+1)z^{-2n} \tag{5}$$

Defining

$$E_0(z) = \sum_{n=-\infty}^{\infty} h(2n)z^{-n}, E_1(z) = \sum_{n=-\infty}^{\infty} h(2n+1)z^{-n} \tag{6}$$

So $H(z)$ can be written as

$$H(z) = E_o(z^2) + z^{-1}E_1(z^2) \tag{7}$$

These representations are valid for whether $H(z)$ is an FIR or IIR; causal or noncausal.
Now suppose that we are given any integer M, so $H(z)$ can be decomposed as [1]

$$\begin{aligned} H(z) = & \sum_{n=-\infty}^{\infty} h(nM)z^{-nM} \\ & + z^{-1} \sum_{n=-\infty}^{\infty} h(nM+1)z^{-nM} \\ & \vdots \\ & + z^{M-1} \sum_{n=-\infty}^{\infty} h(nM+M-1)z^{-nM} \end{aligned} \tag{8}$$

this can be written as

$$H(z) = \sum_{l=0}^{M-1} z^{-l}E_l(z^M) \text{ (Type 1 Polyphase)} \tag{9}$$

where

$$E_l(z) = \sum_{n=-\infty}^{\infty} e_l(n) z^{-n}$$
with
$$e_l(n) \triangleq h(Mn + l),\ 0 \leq l \leq M-1 \tag{10}$$

A variation of (8) is given by [1]

$$H(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_l(z^M) \text{ (Type 2 Polyphase)} \tag{11}$$

Type 2 polyphase components $R_l(z)$ are permutations of $E_l(z)$, i.e. $R_l(z) = E_{M-1-l}(z)$

3.1 Efficient Structures for Decimation and Interpolation Filters

3.1.1 Decimation Filters

Consider the decimation filter with $M = 2$. If $H(z)$ is represented as in (7), then systems can be redrawn as shown in Figure 9(a). By invoking noble identity 1, this can be redrawn as shown in Figure 9(b). This implementation is more efficient than a direct implementation of $H(z)$ (Figure 9(c)). As shown, the direct implementation computes only even numbered output samples, which requires $N + 1$ multiplications and N additions. As time changes from $2n$ to $2n + 1$, the stored signals in the delays change. This means that above computations must be completed in one unit of time. The speed of operation should therefore correspond to $N + 1$ multiplications and N additions *per unit time*. During the odd instants of time, the structure is merely resting. This is inefficient resource utilization.
Now if we consider the polyphase representation given in Figure 9(b), let n_0 and n_1 be the orders of $E_0(z)$ and $E_1(z)$. So $E_l(z)$ requires n_l+1 multiplications and n_l additions. The total cost is again $N + 1$ multipliers and N adders. However, since $E_l(z)$ operates at lower rate, only a total of $(N + 1)/2$ multiplications per *unit time* (MPUs) and $N/2$ additions per *unit time* (APUs) are required. The multipliers and adders in each of the filters $E_0(z)$ and $E_1(z)$ now have two units of time available for their work, and they are continually operative (i.e., no resting time).

3.1.2 Inerpolation Filters

Consider an interpolation filter with $L = 2$. A direct form implementation of $H(z)$ is again inefficient because, at most 50% of the input samples to $H(z)$ are nonzero, which means that at any point in time, only 50% of multipliers of $h(n)$ have nonzero input. So the remaining multipliers are resting. And those multipliers which are not resting are expected to complete their job in *half unit* of time because the outputs of the delay elements will change by that time. A more efficient structure can be obtained by using Type 2 polyphase decomposition.

$$H(z) = R_1(z^2) + z^{-1} R_0(z^2) \tag{12}$$

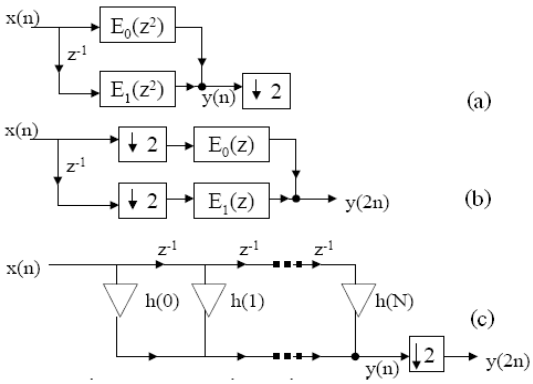

Figure 9: (a) Polyphase representation of decimation filter (b)Using noble identity 1 (c) direct form implementation

This is shown in Figure 10. It can be seen now that $R_l(z)$ are opearting at the input rate, and none of the multipliers are resting. Each multiplier gets one unit of time to finish its task. The complexity of the system is $N+1$ MPUs and $N-1$ APUs. The extra adder following the expander is *not counted* because, the signal $y(n)$ is obtained merely by interlacing $y_0(n)$ and $y_1(n)$.

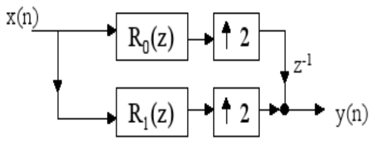

Figure 10: Polyphase representation of an interpolation filter

4 Applications of Multirate Systems

4.1 Subband Coding of Speech and Image Signals

Often, we cencounter signals that are not bandlimited, but still have dominant frequency bands. An example is shown in Figure 11(a).The information in $|\omega| \leq \pi/2$ is not small enough to be discarded. And $x(n)$ cannot be decimated without causing aliasing either. This is because of the small fraction of energy in the high frequency region that prevent to compress the signal.
But if the signal is splitted into two frequency bands by using an analysis bank with responses as shown in Figure : 11(b), then the subband signal $x_1(n)$ has less energy than $x_0(n)$ and so can be encoded with less number of bits.

The reconstruction of the full band signal is done using the expanders and synthesis filters as shown in Figure 12.

So a generalization of this idea can be as: split the signal into two or more subbabnds, decimate each subband signal, and allocate bits for samples in each subband depending on th energy content. In speech coding practice the number of subbands, filter bandwidths and bit allocations are chosen to further exploit

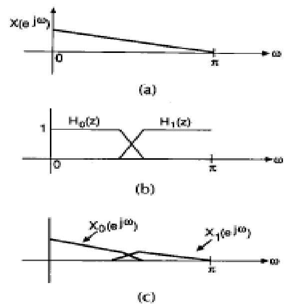

Figure 11: Splitting a signal into subbabnd signals $x_0(n)$ and $x_1(n)$

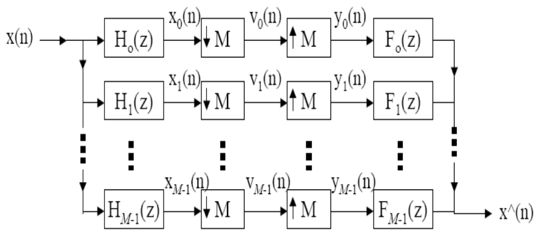

Figure 12: The analysis/synthesis system for subband coding.

the *perceptual properties* of human hearing. the complete analysis synthesis system (Figure 12) is called the *Quadrature Mirror Filter* (QMF) bank.
For image processing, multirate filters have different applications such as multiresolution systems and the Laplace pyramid.

4.2 Transmultiplexers

A complete transmultiplexer is shown in Figure 13. The components $x_k(n)$ of the TDM version can be recovered by separating the consecutive regions of $Y(e^{j\omega})$ with the help of analysis bank and then decimating the outputs. If the synthesis filters $F_k(z)$ and analysis filters $H_k(z)$ are non ideal, then the reconstructed signals $\hat{x}$ has contributions from the desired signal and $x_k(n)$ as well as the cross talk terms $x_l(n)$, $l \neq k$. To reduce the extent of cross talk, $H_k(z)$ and $F_k(z)$ can be designed to have *non overlapping* frequency responses. To reduce the cross talk to an acceptable value requires filters of very high order.

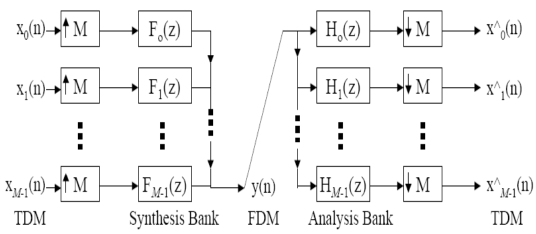

Figure 13: The complete transmultiplexer structure

5 Summary

In this paper, a very brief introduction of Multirate Systems and Filter Banks has been presented. An introduction to the technique "Polyphase Representation" and how it can be used to make multirate systems computationally efficient has also been shown with examples. At the end, some applications of the multirate systems have been discussed.

References

[1] VAIDYANATHAN, P.P.: *Multirate Systems and Filter Banks, 2nd Ed*, Prentice Hall , 1993.

[2] VAIDYANATHAN, P.P.: *Multirate Digital Filters, Filter Banks, Polyphase Networks, and Applications: A Tutorial*, Proceedings of the IEEE, Vol. 78, No. 1, January 1990.