Series Expansion with Wavelets

Advanced Signal Processing 2 - 2007

Reinisch Bernhard
Teichtmeister Georg

Introduction

- Series expansion
- Fourier Series: Either periodic or bandlimited signals
- Timedomain: No frequency information
- Fourierdomain: No time information
- Is there something between?

Contents

- Basics of signal representation
- Wavelets
  - Haar wavelet
  - Multiresolution analysis
  - Construction of the Sinc - Wavelet
- Wavelets derived from iterated filter banks
  - Haar case, Sinc case, general construction
- Wavelet series and its properties
- Practical outlook (image processing)

Recap of Series expansion

- Signals are points in a Vectorspace
- Time-Domain: Basis functions are infinite short impulses
- Signals can be projected onto other basis functions

$$f(t) = \sum_{k=-\infty}^{\infty} \langle \varphi_k(u), f(u) \rangle \varphi_k(t)$$

$$\langle \varphi_k(u), f(u) \rangle = \int_{-\infty}^{\infty} \varphi_k^*(u) f(u) du$$

Possible Basis Functions

- Fourierseries
  - periodic

$$f(t)=\sum_{k=-\infty}^{\infty} F[k]e^{j(2\pi kt)/T}$$

- Fouriertransform
  - bandlimited

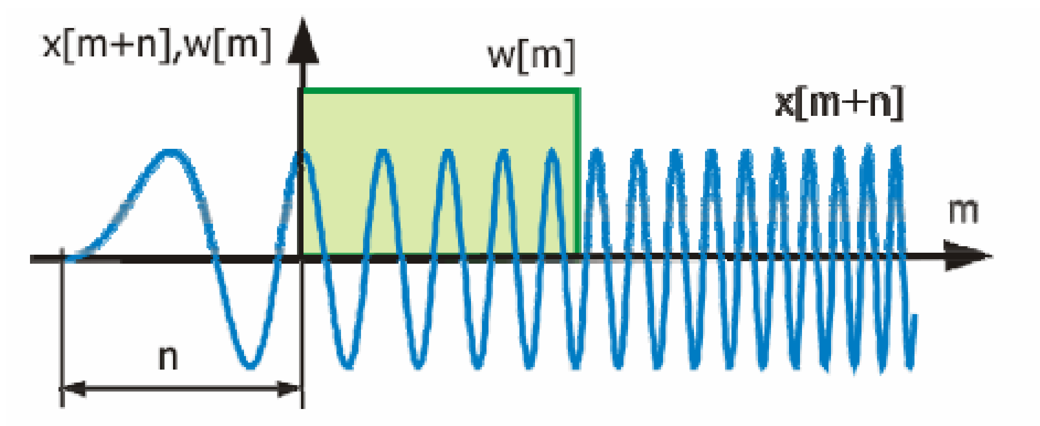
$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

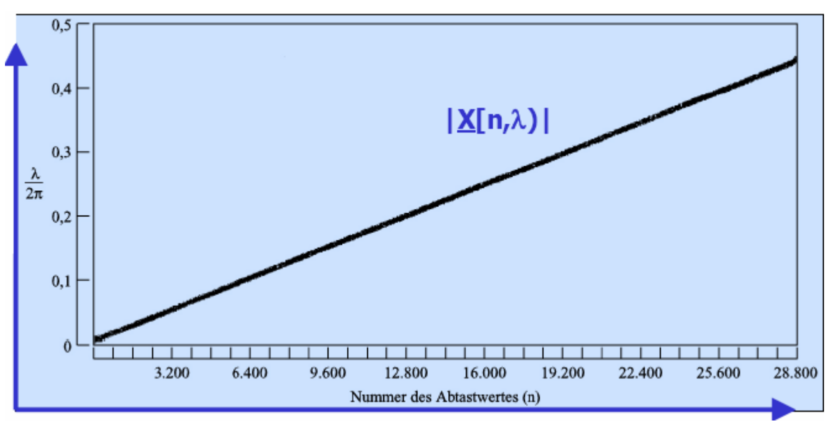
- STFT
  - Infinite set of Fourier Transforms
- Piecewise Fourier Series
- Wavelets

Short Time Fourier Transform

- Window Signal
- Compute the Fourier Transform
- Shift window and repeat

⇒ Spectrogram, Periodogram



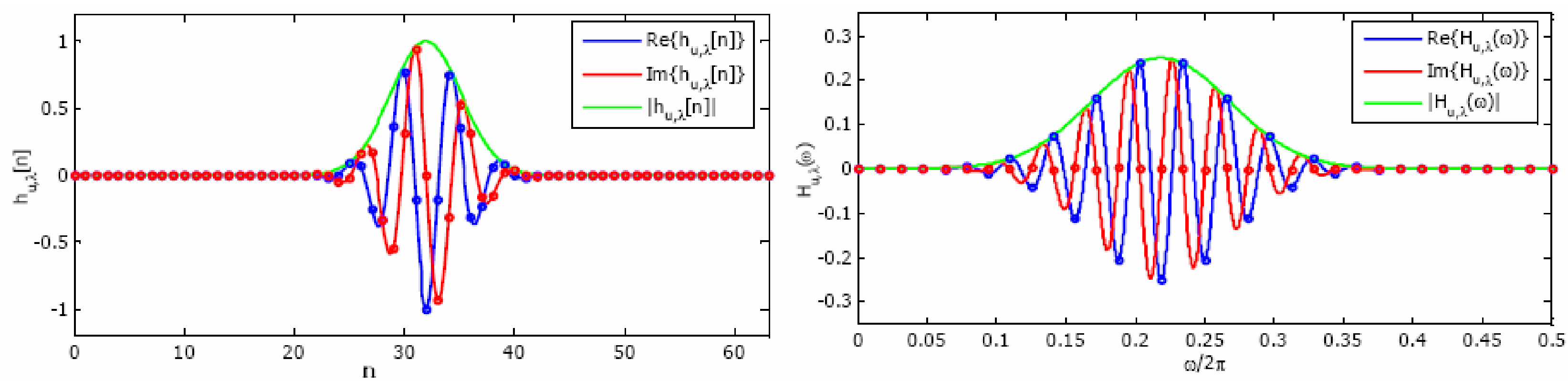


Time and Frequency Resolution

- Window has Energydistribution in both: Frequency (σ_ω) and Time (σ_n).
- Uncertainty principle:

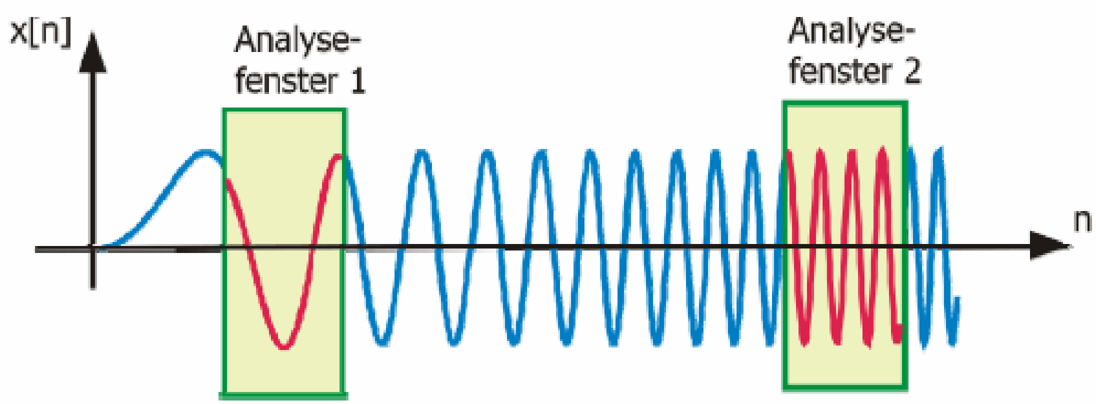
$$\sigma_\omega \sigma_n \geq \frac{1}{2}$$

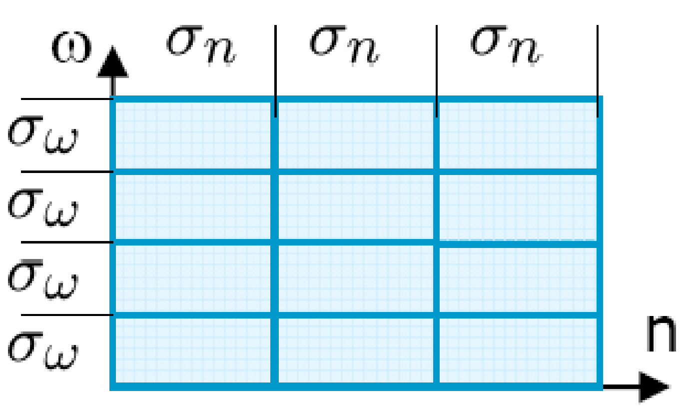
- Optimality is only reached by Gaussian window



STFT T/F-Resolution

- Constant over Time and Frequency





Piecewise Fourierseries

- Fourier Series with non-overlapping rectangular windows in time and periodic expansion
- Why?
  - Overlapping windows are redundant information
  - Good Time Resolution
  - Representation of arbitrary functions
- Bad Frequency Resolution
- Errors at boundaries

Desired Features of Basis Functions

- Simple characterization
- Localization Properties in Time and Frequency
- Invariance under certain operations
- Smoothness properties
- Moment properties

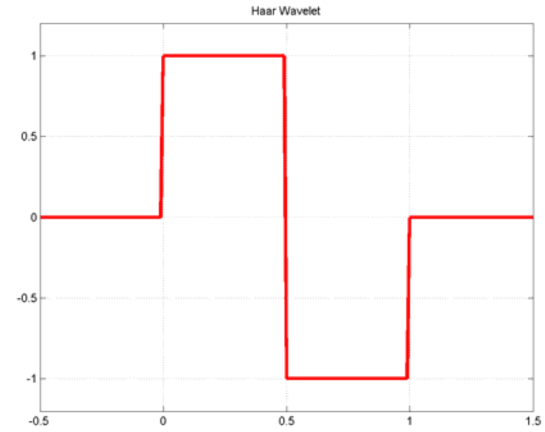
Haar - Expansion

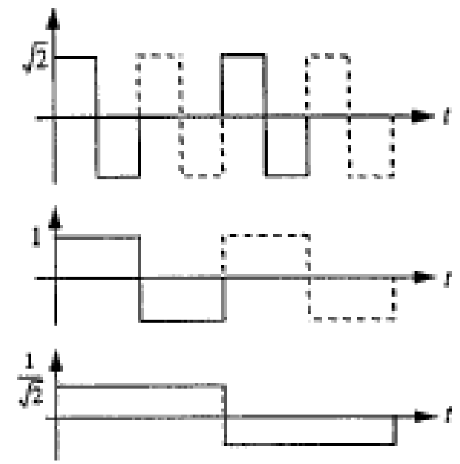
- Simplest Wavelet Expansion

- Scaled and shifted Wavelets:

$$\phi_{m,n}(t) = 2^{-m/2}\phi(2^{-m}t - n)$$

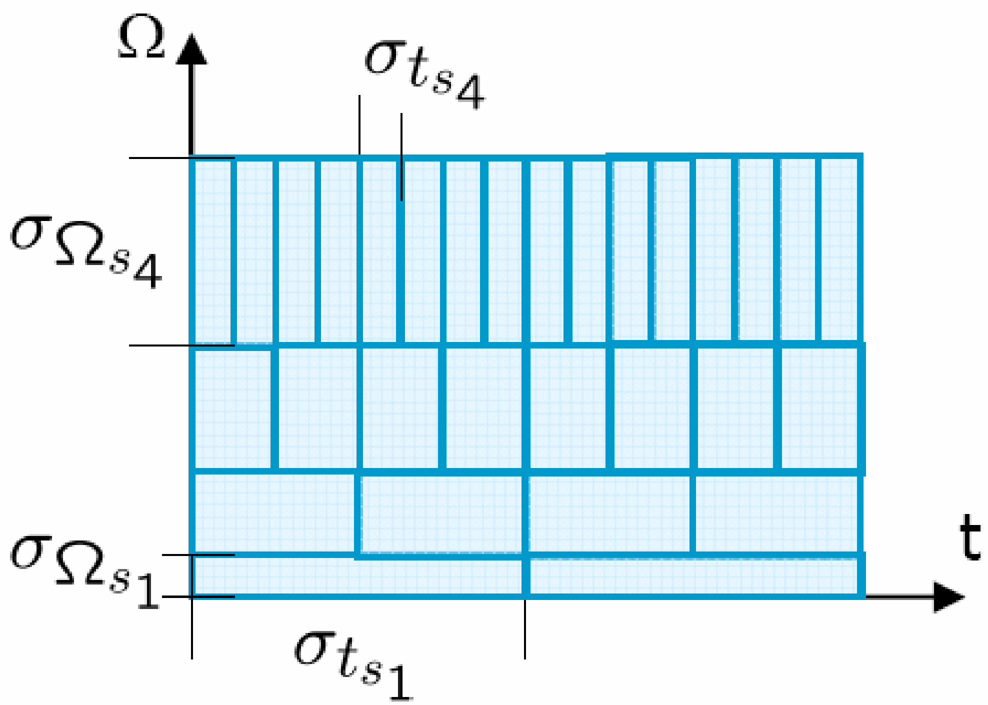
- m … Scale
- n … Timeshift





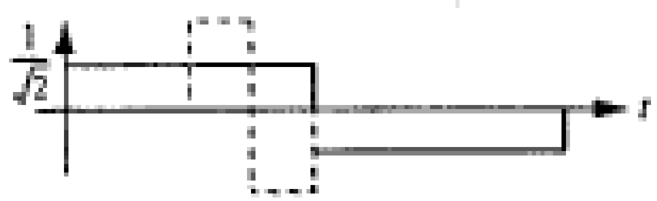
Dyadic Tiling

- Resolution depends on Frequency now
- $\phi_{m,n}(t) = 2^{-m/2}\phi(2^{-m}t - n)$



Orthonormal Basis for L_2?

- Two wavelets on the same Scale have no common support
- Shorter wavelet always averages to zero
- Shifting so that jump matches, is not possible



Proof: Definitions

- Consider functions which are constant on

$$[n2^{-m_0},(n+1)2^{-m_0}]$$

- and have finite support on

$$[-2^{m_1},2^{m_1}]$$

- Can approximate L_2 arbitrarily well
- We call it $f^{(-m_0)}(t)$

Proof: Scaling function

- The scaling function

$$\varphi_{-m_0,n}(t) = \begin{cases} 2^{\frac{m_0}{2}} & n2^{-m_0} \le t < (n+1)2^{-m_0} \\ 0 & otherwise \end{cases}$$

- Approximating the piecewise constant function

$$f^{(-m_0)}(t) = \sum_{n=-N}^{N-1} f_n^{(-m_0)} \varphi_{-m_0,n}$$

$$N = 2^{m_0+m_1}$$

$$f_n^{(-m_0)} = 2^{\frac{-m_0}{2}} f^{(-m_0)}(n2^{-m_0})$$

Proof: Illustration

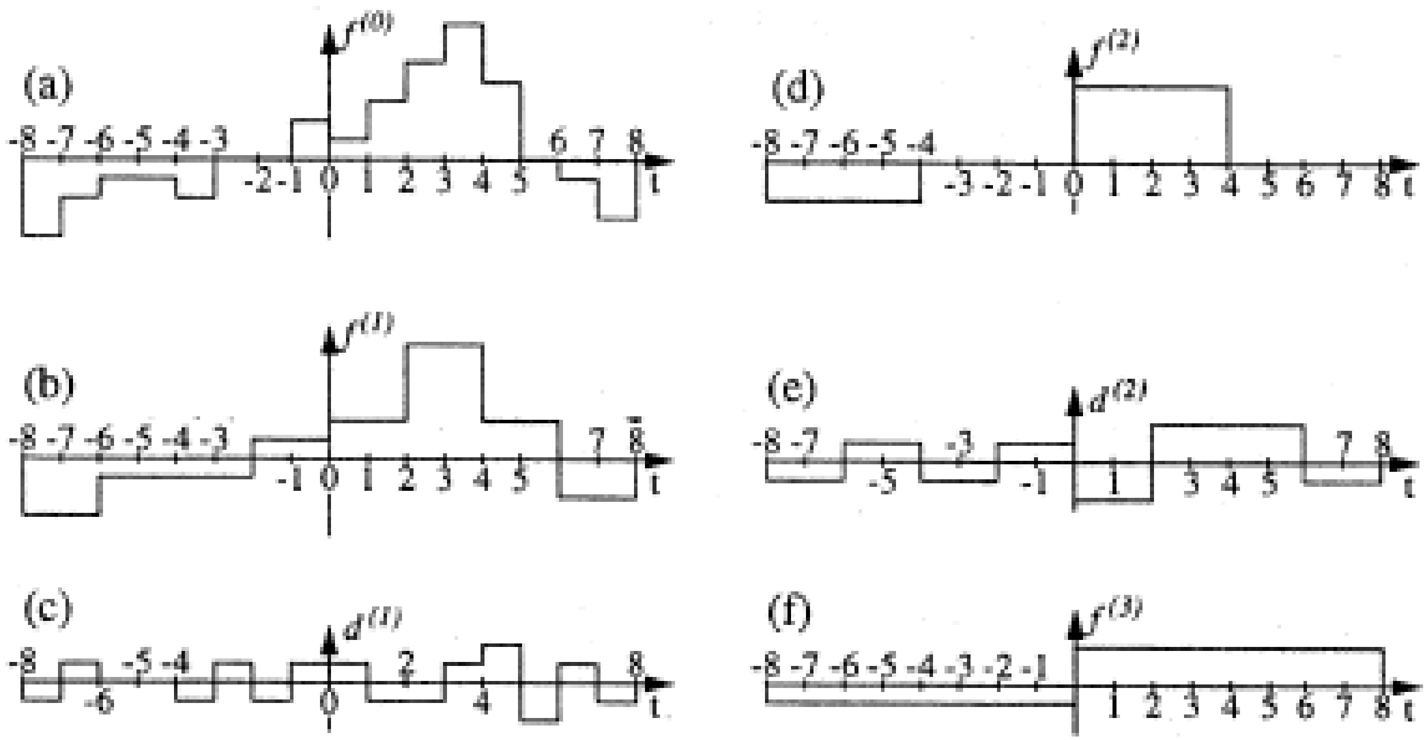


Figure 4.3 Haar wavelet decomposition of a piecewise continuous function. Here, $m_0 = 0$ and $m_1 = 3$. (a) Original function $f^{(0)}$. (b) Average function $f^{(1)}$. (c) Difference $d^{(1)}$ between (a) and (b). (d) Average function $f^{(2)}$. (e) Difference $d^{(2)}$. (f) Average function $f^{(3)}$.

Proof: Keystep

- Examination of two adjacent Intervals

$$\left[2n2^{-m_0},(2n+1)2^{-m_0}\right)\text{and}\left[(2n+1)2^{-m_0},(2n+2)2^{-m_0}\right)$$

- Now $f^{(-m_0)}(t)$ can be expressed as

$$f_{2n}^{(-m_0)}\varphi_{-m_0,2n}(t)+f_{2n+1}^{(-m_0)}\varphi_{-m_0,2n+1}(t)$$

- For m_0=0, n=1, this means

$$[2,3)\ \text{and}\ [3,4)$$

$$f_2^{(0)}\varphi_{0,2}(t)+f_3^{(0)}\varphi_{0,3}(t)=2\varphi_{0,2}(t)+3\varphi_{0,3}(t)$$

Proof: Average and Difference

The function $f^{(-m_0)}(t)$ can also be expressed as the average

$$\frac{f_{2n}^{(-m_0)} + f_{2n+1}^{(-m_0)}}{2} \sqrt{2}\varphi_{-m_0+1,n}(t)$$

and the difference

$$\frac{f_{2n}^{(-m_0)} - f_{2n+1}^{(-m_0)}}{2} \sqrt{2}\phi_{-m_0+1,n}(t)$$

over two intervals

Proof: Coefficients

- With

$$f_n^{(-m_0+1)} = \frac{1}{\sqrt{2}}\left(f_{2n}^{(-m_0)} + f_{2n+1}^{(-m_0)}\right)$$

$$d_n^{(-m_0+1)} = \frac{1}{\sqrt{2}}\left(f_{2n}^{(-m_0)} - f_{2n+1}^{(-m_0)}\right)$$

- We get

$$f_{2n}^{(-m_0)}\varphi_{-m_0,2n}(t) + f_{2n+1}^{(-m_0)}\varphi_{-m_0,2n+1}(t) =$$

$$f_n^{(-m_0+1)}\varphi_{-m_0+1,n}(t) + d_n^{(-m_0+1)}\phi_{-m_0+1,n}(t)$$

Proof: Finalization

- Applying all the things we can write

$$f^{(-m_0)}(t) = f^{(-m_0+1)}(t) + d^{(-m_0+1)}(t) =$$

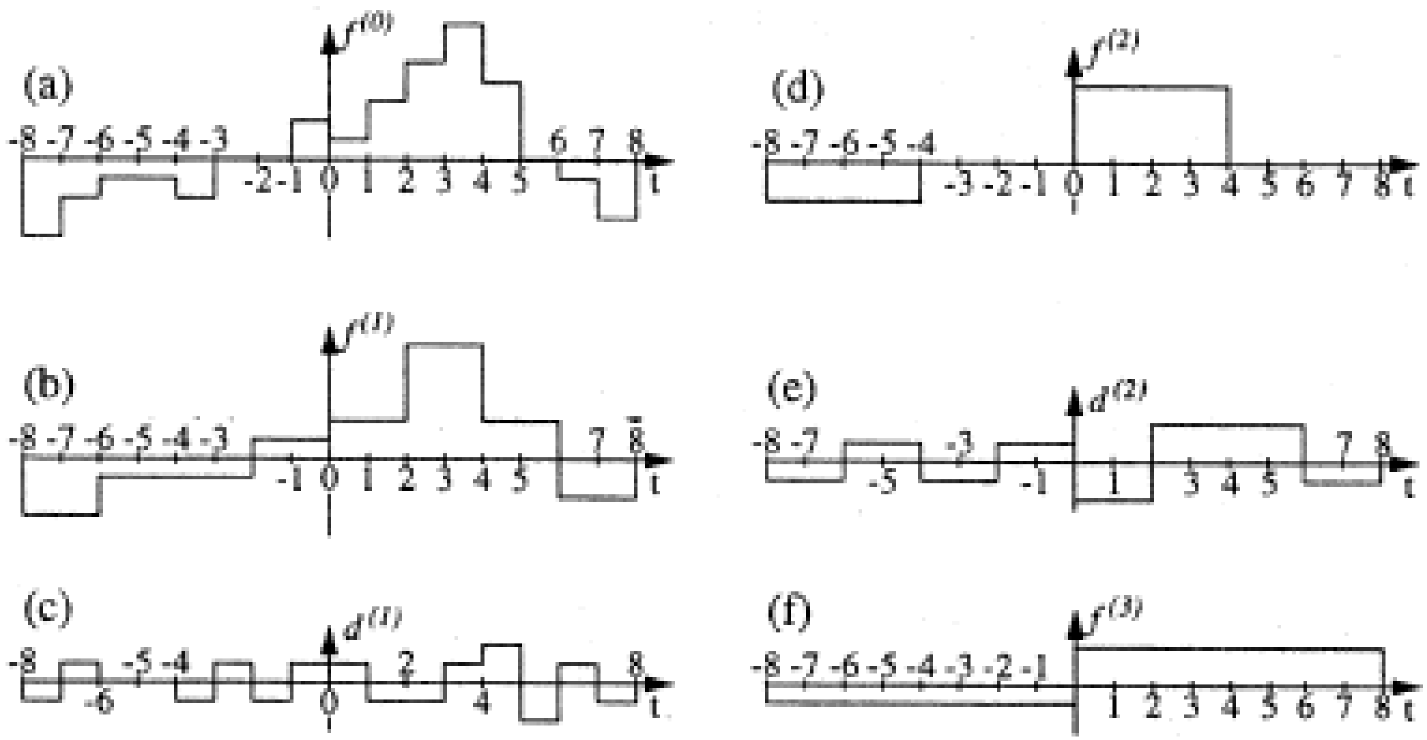
$$= \sum_{n=-N/2}^{N/2-1} f_n^{(-m_0+1)} \varphi_{-m_0+1,n}(t) + \sum_{n=-N/2}^{N/2-1} d_n^{(-m_0+1)} \phi_{-m_0+1,n}(t)$$

- Repeating the Average/Difference scheme for higher scales leads to

$$f^{(-m_0)}(t) = f^{(m1)}(t) + \sum_{m=-m_0+1}^{m_1} \sum_{n=-2^{m_1-m}}^{2^{m_1-m}-1} d_n^{(m)} \phi_{m,n}(t) =$$

$$= \sum_{m=-m_0+1}^{m_1+M} \sum_{n=-2^{m_1-m}}^{2^{m_1-m}-1} d_n^{(m)} \phi_{m,n}(t) + \varepsilon_M$$

Proof: Illustration



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Multiresolution

- Successive approximation
- Coarse approximation + added details
- Coarse and detail subspace are orthogonal
- Leads to self-similar Wavelets in Scale
- Useful for applications

Axiomatic Definition (1)

- Sequence of embedded closed subspaces

$$\ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$$

- Upward Completeness

$$\bigcup_{m \in Z} V_m = L_2(R)$$

- Downward Completeness

$$\bigcap_{m \in Z} V_m = \{0\}$$

Axiomatic Definition (2)

- Scale Invariance

$$f(t) \in V_m \Leftrightarrow f(2^m t) \in V_0$$

- Shift Invariance

$$f(t) \in V_0 \Rightarrow f(t-n) \in V_n \qquad \forall n \in Z$$

- Existance of a orthonormal Basis
  - Non-orthogonal Basis can be orthogonalized

Orthogonal Complements

- V_m is a subspace of V_{m-1}
- We define W_m the orthogonal subset of V_m in V_{m-1}
- $V_{m-1} = V_m \oplus W_m$
- V_m is the space of the scaling functions
- W_m the space of the wavelets
- By repeating we get

$$L_2(R) = \bigoplus_{m \in Z} W_M$$

Constructing the Sinc Wavelet

- Now the scaling functions will be the space of bandlimited functions
- V_0 is bandlimited between [-π, π], V_{-1} between [-2π, 2π]
- W_0 the functions bandlimited to [-2π, -π] combined with [π, 2π]
- $$V_{-1} = V_0 \oplus W_0$$

Scaling function

- The scaling function is given by

$$\varphi(t) = \frac{\sin \pi t}{\pi t}$$

Representation of φ

- V_0 belongs to V_{-1}
- φ(t) can be represented by basis functions of V_{-1}

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)$$

$$\|g_0[n]\| = 1;\ g_0[n] = \sqrt{2} \langle \varphi(2t-n), \varphi(t) \rangle$$

- Without proof

$$g_1[n] = (-1)^n g_0[-n+1]$$

$$\phi(t) = \sqrt{2} \sum_{n \in Z} g_1[n] \varphi(2t-n)$$

Construction Kernel

- g_0 is given by

$$g_0[n] = \frac{1}{\sqrt{2}} \frac{\sin(\pi n/2)}{\pi n/2}$$

$$G_0(e^{j\omega}) = \begin{cases} -\sqrt{2}e^{-j\omega} & -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \\ 0 & otherwise \end{cases}$$

- And finally the wavelet

$$\phi(t) = \frac{\sin(\pi t/2)}{\pi t/2} \cos(3\pi t/2)$$

Sinc Wavelet: Illustration

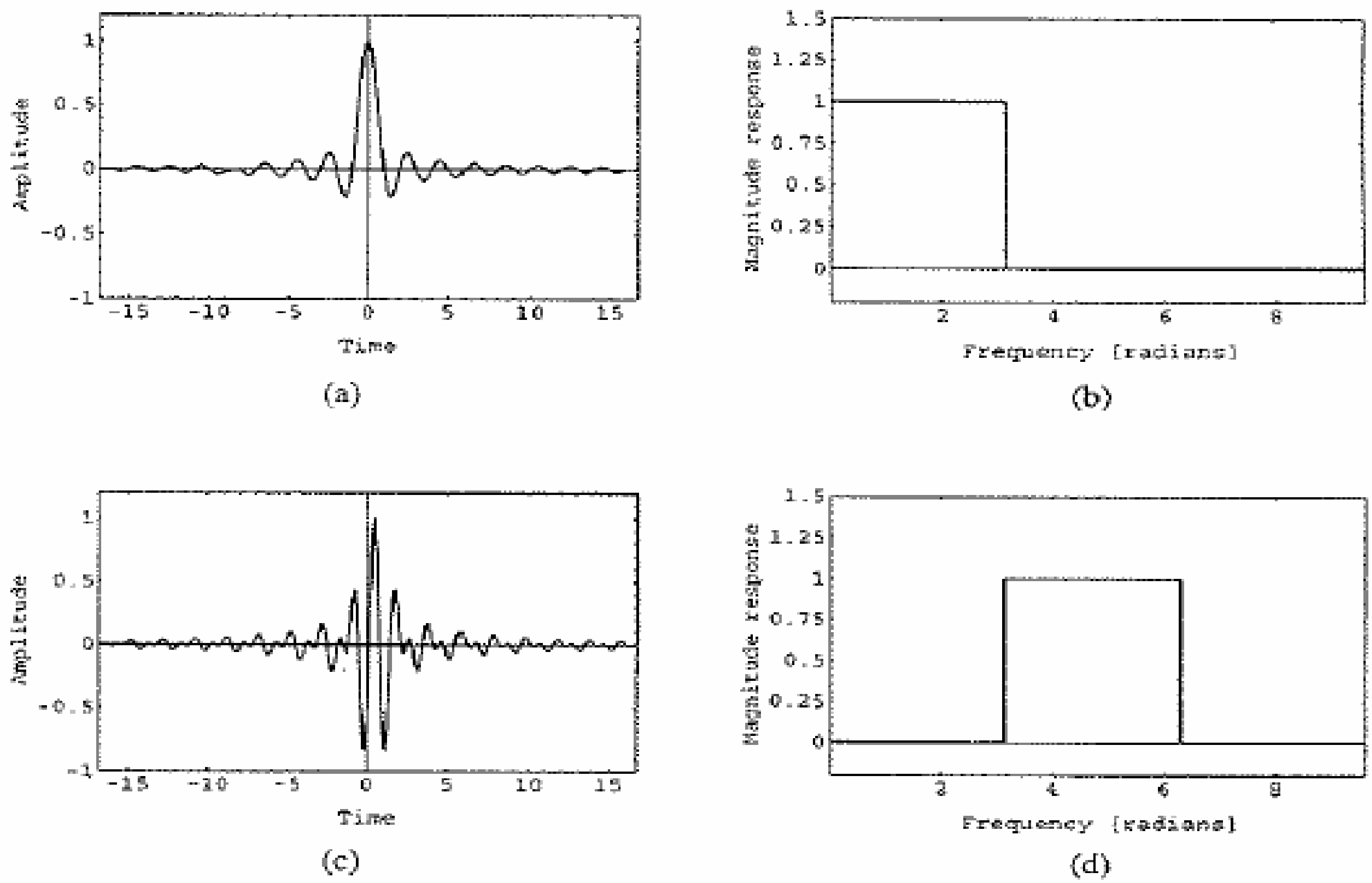


Figure 4.6 Scaling function and the wavelet in the sinc case. (a) Scaling function $\varphi(t)$. (b) Fourier transform magnitude $|\Phi(\omega)|$. (c) Wavelet $\psi(t)$. (d) Fourier transform magnitude $|\Psi(\omega)|$.

Iterated filter banks

- Until now we constructed wavelets by scaling and shifting of orthonormal function families
  - Based on multiresolution analysis

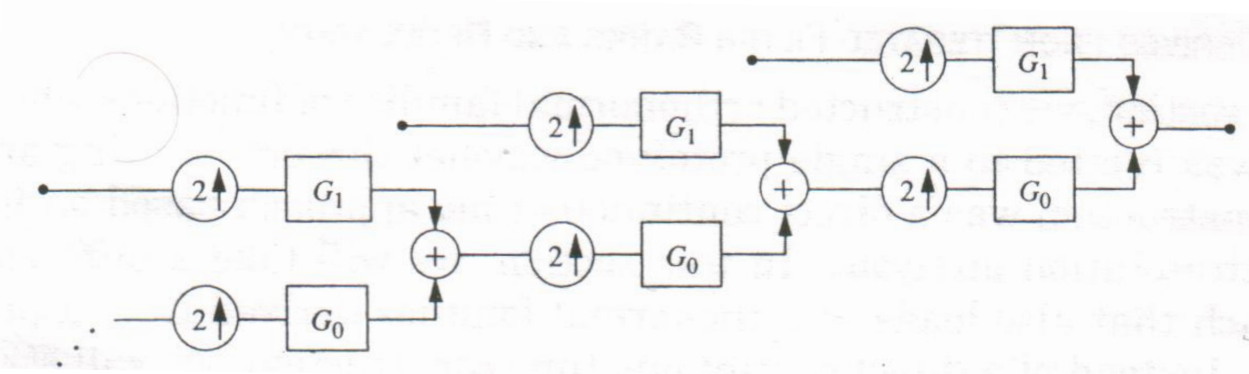
- Different approach by filter banks
  - Iteration leads to a wavelet
  - Key proberties
    - regularity
    - degree of regularity

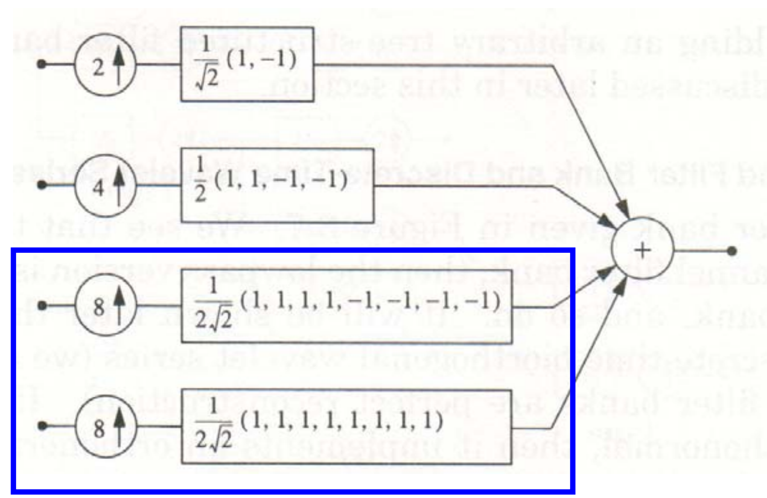
Haar case

- Low- and Highpass

$$g_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]; \; g_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]$$

- Iterate the filter bank on the lowpass channel





- Multirate signal processing results

Haar case

- size-8 discrete Haar transform

$$g_0^{(i)}[n] = \begin{cases} 2^{-i/2} & n = 0, \ldots, 2^i - 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$g_1^{(i)}[n] = \begin{cases} 2^{-i/2} & n = 0, \ldots, 2^{i-1} - 1, \\ -2^{-i/2} & n = 2^{i-1}, \ldots, 2^i - 1, \\ 0 & \text{otherwise}, \end{cases}$$

- Number of coefficients growth exponentially
- Continuous time function

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n] \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i},$$

$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n] \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i}.$$

- Length bounded, piecewise constant

Sinc case (1)

- Impulse responses (low and highpass filter)

$$g_0[n] = \frac{1}{\sqrt{2}} \frac{\sin(\pi/2n)}{\pi/2n};\ g_1[n] = (-1)^n g_0[-n+1]$$

- Fourier transform

$$G_0(e^{j\omega}) = \begin{cases} \sqrt{2} & -\frac{\pi}{2} \le \omega \le \frac{\pi}{2}, \\ 0 & \text{otherwise,} \end{cases} \qquad G_1(e^{j\omega}) = \begin{cases} -\sqrt{2}e^{-j\omega} & \omega \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi], \\ 0 & \text{otherwise,} \end{cases}$$

- Now consider the iterated filter bank
  - Upsampling filter impulse
  - Emulate the Haar construction with $g_0[n]$, $g_1[n]$
  - And define a scaling function

Sinc case (2)

Fourier transform of $\varphi^{(i)}(t)$

$$\Phi^{(i)}(\omega) = 2^{-i/2} G_0^{(i)}(e^{-j\omega/2^i}) e^{-j\omega/2^{i+1}} \frac{\sin(\omega/2^{i+1})}{\omega/2^{i+1}}$$

where :

$$G_0^{(i)}(e^{j\omega}) = G_0(e^{j\omega}) G_0(e^{j2\omega}) \ldots G_0(e^{j2^{i-1}\omega})$$

short :

$$M_0(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega})$$

we can rewrite :

$$\Phi^{(i)}(\omega) = \left[\prod_{k=1}^{i} M_0\left(\frac{\omega}{2^k}\right)\right] e^{-j\omega/2^{i+1}} \frac{\sin(\omega/2^{i+1})}{\omega/2^{i+1}}$$

- For further analysis: important part is in the brackets
- This product is $2^i 2\pi$ periodic -> in the end it‘s only a perfect lowpass (sinc scaling function)

Sinc case (3)

- Cumbersome way
- But we have gained a more general construction
- The key is the infinite product
  - Does this product converge and to what
  - Converge to what kind of scaling function

Iterated filter banks cont. (1)

- General construction
  - Two channel orthogonal filter bank
  - $g_0[n]$, $g_1[n]$ are low- and highpass filter
  - Iterate on the branch of the lowpass filter and process this to infinity
  - Express the two filters after i-steps
  - Multirate conclusions
    - „Filtering with $G_i(z)$ followed by upsampling by 2 is equivalent to upsampling by 2 followed by filtering wih $G_i(z^2)$„

Iterated filter banks cont. (2)

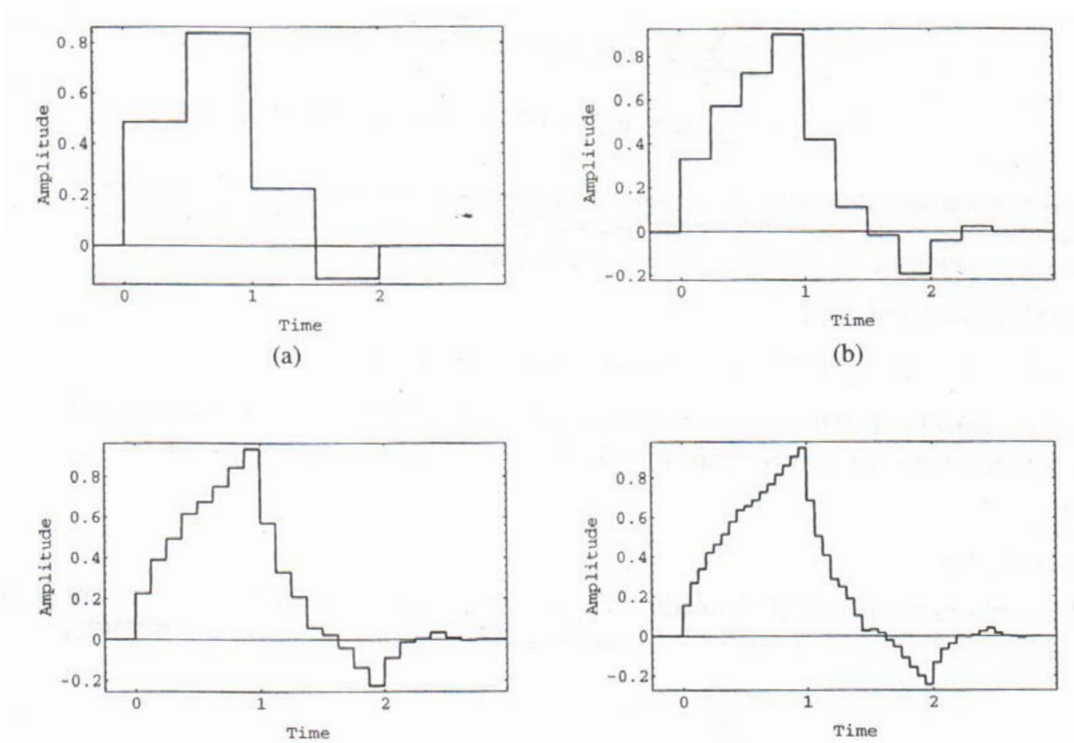
$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0\left(z^{2^k}\right),$$

$$G_1^{(i)}(z) = G_1(z^{2^{i-1}}) \prod_{k=0}^{i-2} G_0\left(z^{2^k}\right), \qquad i = 1, 2, \ldots$$

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \quad n/2^i \le t < \frac{n+1}{2^i},$$

$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n], \quad n/2^i \le t < \frac{n+1}{2^i}.$$

- Discrete time iterated filters combined with the continuous time functions
- Normalization and rescaling
- Graphical function
  - piecewise constant
  - halving the intervall



Iterated filter banks cont. (3)

- Fourier domain act as above
- In the iteration scheme we are interesting in convergence

$$\varphi(t) = \lim_{i\to\infty} \varphi^{(i)}(t),$$
$$\psi(t) = \lim_{i\to\infty} \psi^{(i)}(t).$$

$$\Phi(\omega) = \lim_{i\to\infty} \Phi^{(i)}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right),$$
$$\Psi(\omega) = \lim_{i\to\infty} \Psi^{(i)}(\omega) = M_1\left(\frac{\omega}{2}\right)\prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right),$$

- This will lead us to regularity discussion

Regularity

- The existence of the limit are critical conditions
  - Limits exist if $g_0[n]$ are regular
  - Regular filter leads through iteration to a scaling function with some degree of smoothness (regularity)
  - But not only convergence is sufficient we need also L_2 convergence to build orthonormal bases
  - A lot of sufficient conditions, different approaches

Wavelet series and properties

- Enumeration of some general properties of basis functions

$$f(t) = \sum_{m,n \in Z} F[m,n]\psi_{m,n}(t)$$

$$F[m,n] = \left\langle \psi_{m,n}(t), f(t) \right\rangle = \int_{-\infty}^{+\infty} \psi_{m,n}(t), f(t)dt$$

- Wavelet
  - Linearity, Shift, Dyadic sampling and time frequency tiling, Scaling, Localization, decay properties

Linearity

suppose operator T

$$T[f(t)] = F[m,n] = \langle \psi_{m,n}(t), f(t) \rangle$$

then for any $a, b \in R$

$$T[a\, f(t) + b\, g(t)] = aT(f(t)) + bT(g(t))$$

- The wavelet series is linear. The proof follows from the linearity of the inner product

Shift

- For Fourier transform
  - pair: f(t), F(ω) … f(t-tau), $e^{-j\omega tau}$ F(ω)
- Now for the wavelet series

$$F'[m,n] = \int_{-\infty}^{+\infty} \psi_{m,n}(t) f(t-\tau) dt$$

$$F'[m,n] = \int_{-\infty}^{+\infty} 2^{-m/2} \psi\left(2^{-m} t - n + 2^{-m} \tau\right) f(t) dt$$

$$2^{-m} \tau \in Z \text{ or } \tau = 2^{m} k, k \in Z$$

$$f(t - 2^{m} k) \leftrightarrow F\left[m', n - 2^{m-m'} k\right], m' < m$$

Scaling

- For Fourier transform
  - pair: f(t), F(ω) … f(at), 1/a*F(ω/a)

$$F'[m,n] = \int_{-\infty}^{+\infty} \psi_{m,n}(t)\, f(at)\, dt$$

$$F'[m,n] = 1/a \int_{-\infty}^{+\infty} 2^{-m/2} \psi\left(\frac{2^{-m} t}{a} - n\right) f(t)\, dt$$

$$a = 2^{-k}, k \in Z$$

$$f(2^{-k} t) \leftrightarrow 2^{k} F[m-k, n]$$

Dyadic sampling and time frequency tiling

- It is important to locate the basis functions in the time-frequency plane
- sampling in time, at scale m, with period 2^m

$$\psi_{m,n}(t) = \psi_{m,0}\left(t - 2^m n\right)$$

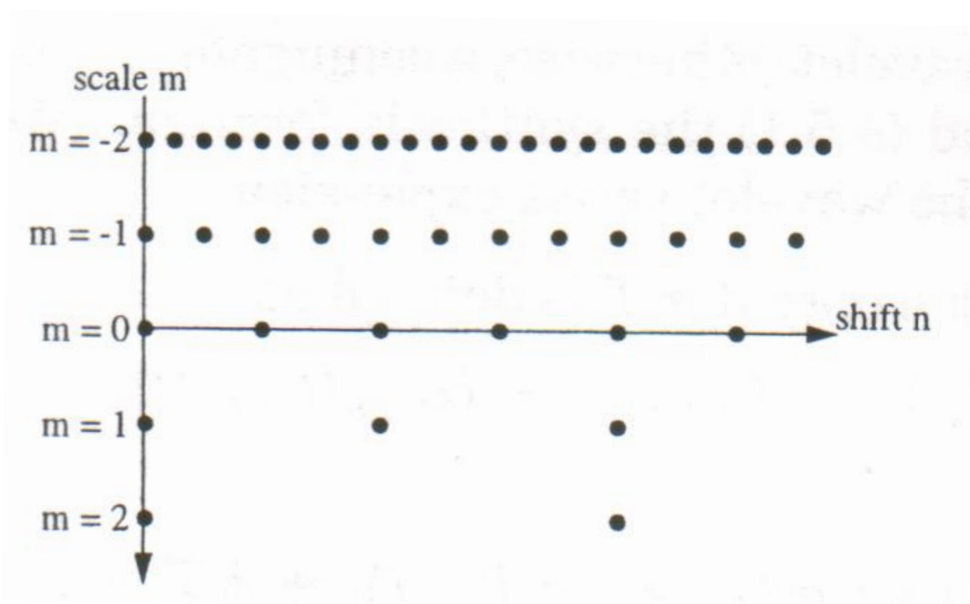
- The frequency is the inverse of scale, we find if the wavelet ist centered around ω_0 then:

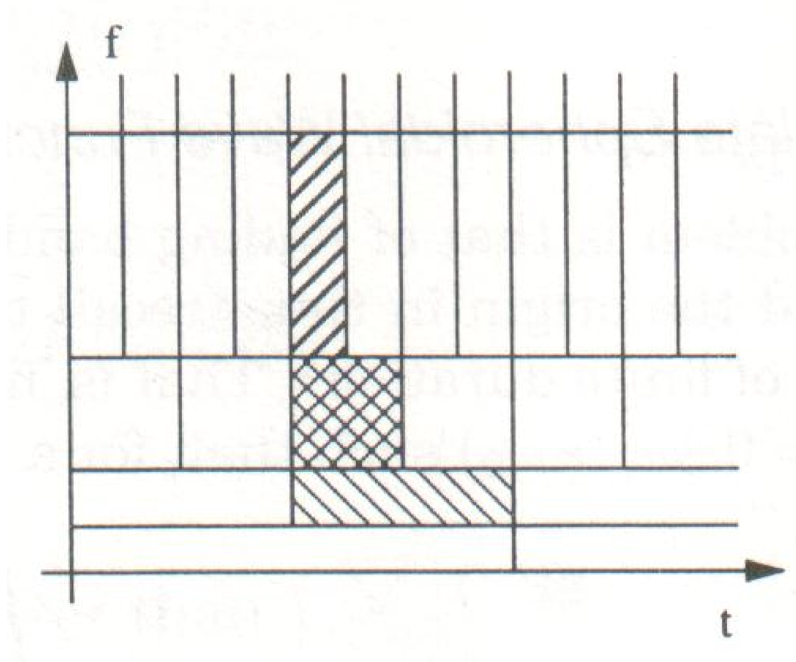
$\Psi_{m,n}(\omega)$ is centered around $\omega_0 / 2^m$

- This leads to dyadic sampling of time frequency plane

Dyadic sampling

- The dots indicate the center of the wavelets
- The scale axis is logarithmic





Time localization (1)

- Suppose we are interested in the signal around $t=t_0$
- Which values of F[m,n] carry information about signal f(t) at t_0 => $f(t_0)$
- Suppose wavelet ψ(t) is supported on the interval $[-n_1, n_2]$
- $\Psi_{m,0}(t)$ is supported on $[-n_1 2^m, n_2 2^m]$
- $\Psi_{m,n}(t)$ is supported on $[(-n_1+n)2^m, (n_2+n)2^m]$

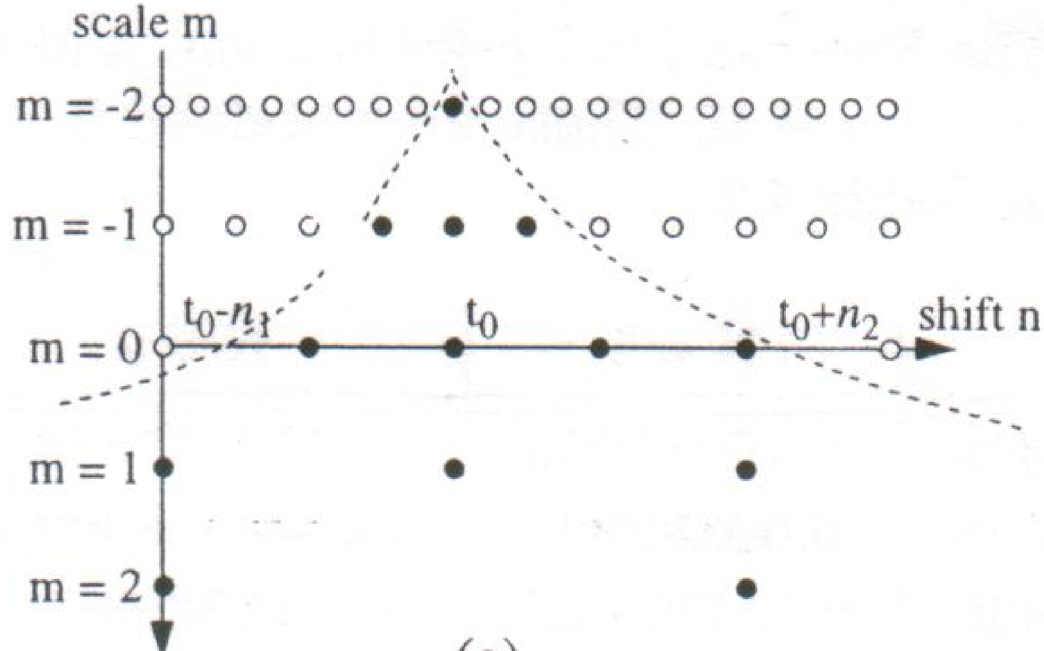
Time localization (2)

- At scale m, wavelet coefficients with index n satisfy

$$(-n_1+n)2^m \le t_0 \le (n_2+n)2^m$$

can be rewritten

$$2^{-m}t_0 - n_2 \le n \le 2^{-m}t_0 - n_1$$



(a)

Frequency localization (1)

- Suppose now in localization, but now in frequency domain

$$\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n)$$

the Fourier transform is

$$2^{m/2}\Psi(2^{m}\omega)e^{-j2^{m}n\omega}$$

$$F[m,n] = \int_{-\infty}^{+\infty}\psi_{m,n}(t)f(t)dt$$

$$F[m,n] = \frac{1}{2\pi}2^{m/2}\int_{-\infty}^{+\infty}F(\omega)\Psi(2^{m}\omega)e^{j2^{m}n\omega}d\omega$$

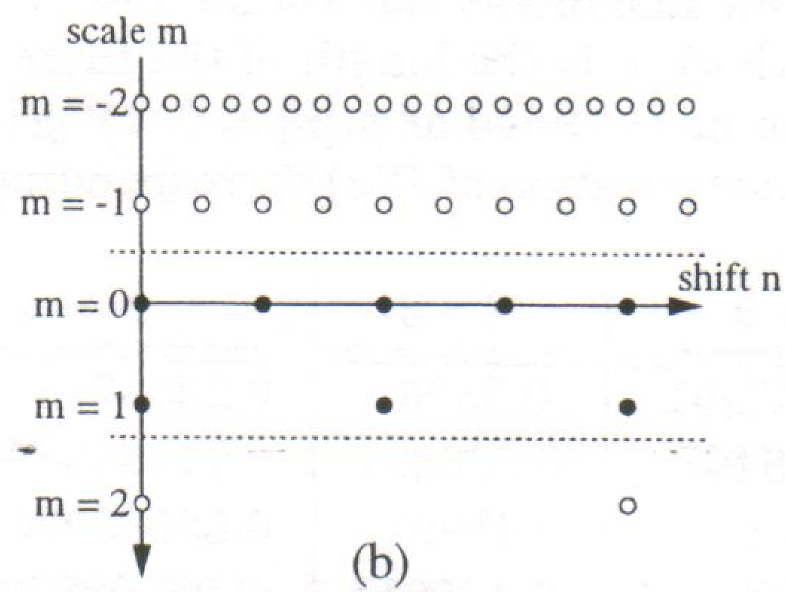
Frequency localization (2)

- Suppose that the wavelet vanishes in the Fourier domain outside the region $[\omega_{min},\omega_{max}]$
- At specific scale m, the support of $\Psi_{m,n}(\omega)$ will be $[\omega_{min}/2^m,\omega_{max}/2^m]$
- Therefore, a frequency component ω_0 incluences at scale m

$$\frac{\omega_{\min}}{2^m} \leq \omega_0 \leq \frac{\omega_{\max}}{2^m}$$

rewrite

$$\log_2\left(\frac{\omega_{\min}}{\omega_0}\right) \leq m \leq \log_2\left(\frac{\omega_{\max}}{\omega_0}\right)$$



Decay properties

- Fourier series can be used to characterize the regularity of a signal (decay of the transfrom coefficients)
  - Global regularity
- The wavelet transform can be used in a similiar way
  - Local regularity

Multidimensional wavelets

- A way to build multidimensional wavelets is to use tensor products of their one dimensional counterparts
  - This will lead to different „mother“ wavelets
  - Scale changes are now represented in matrices
  - offers diagonal scaling but is also more restricted

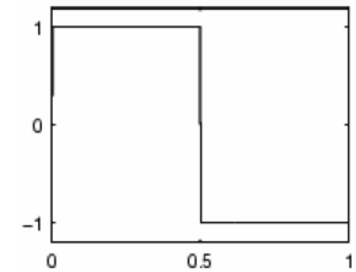
Practical aspects

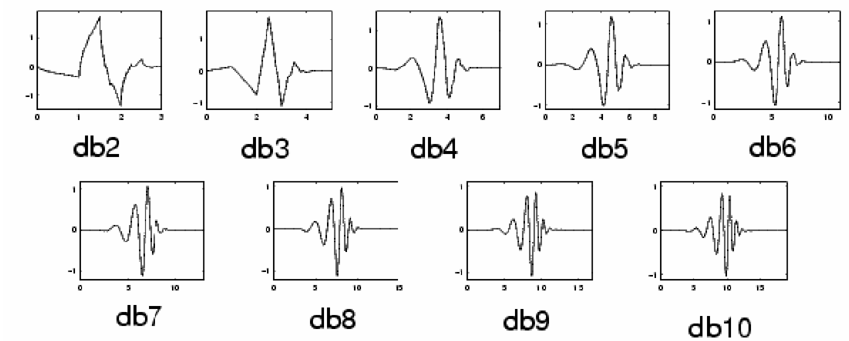
- Wavelets in matlab

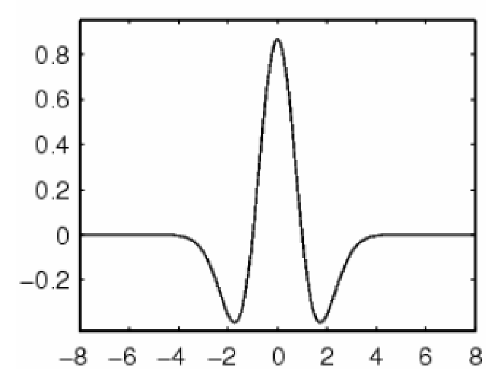
- Images
  - Image compression
  - Edge detection
  - De-noising

Matlab

- Matlab wavelet toolbox
  - Command line
    - Help wavelet
  - Gui tool (wavemenu)
    - 1D wavelets analysis
    - 2D wavelets analysis
    - De-noising
    - Image Fusion
    - Compression

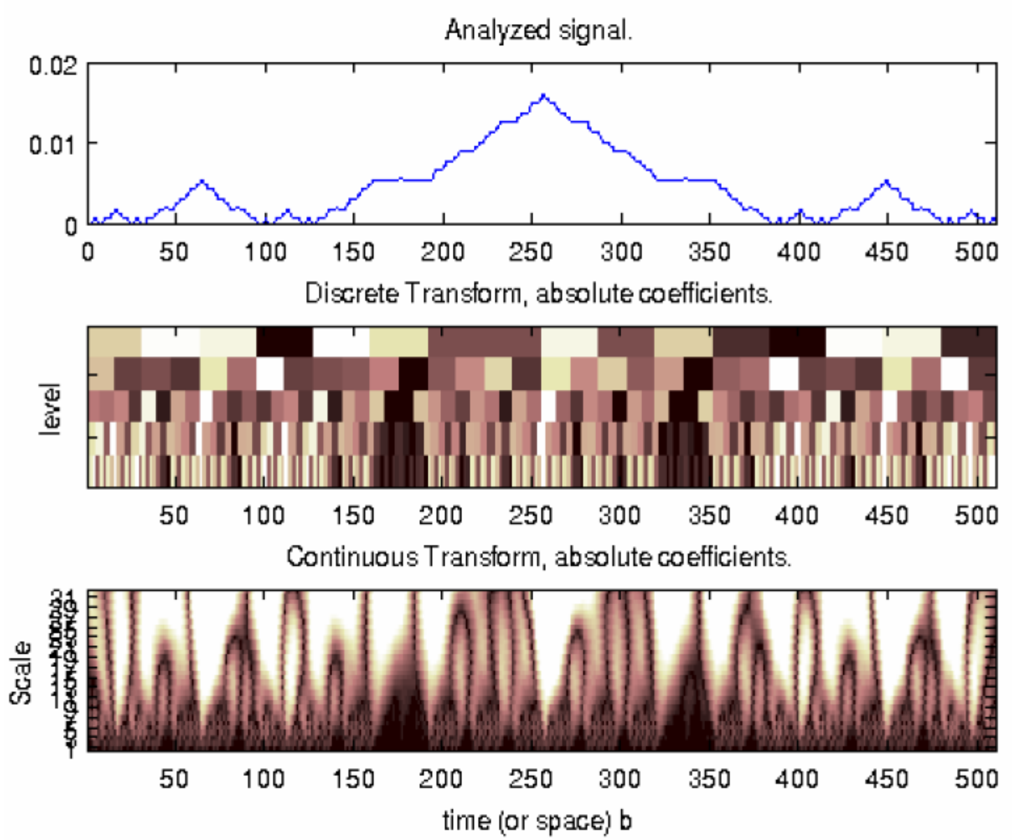






Example 1D wavelet transform

- Discrete/continuous wavelet transform
- coefs = cwt(S, SCALES, ‚wname‘)

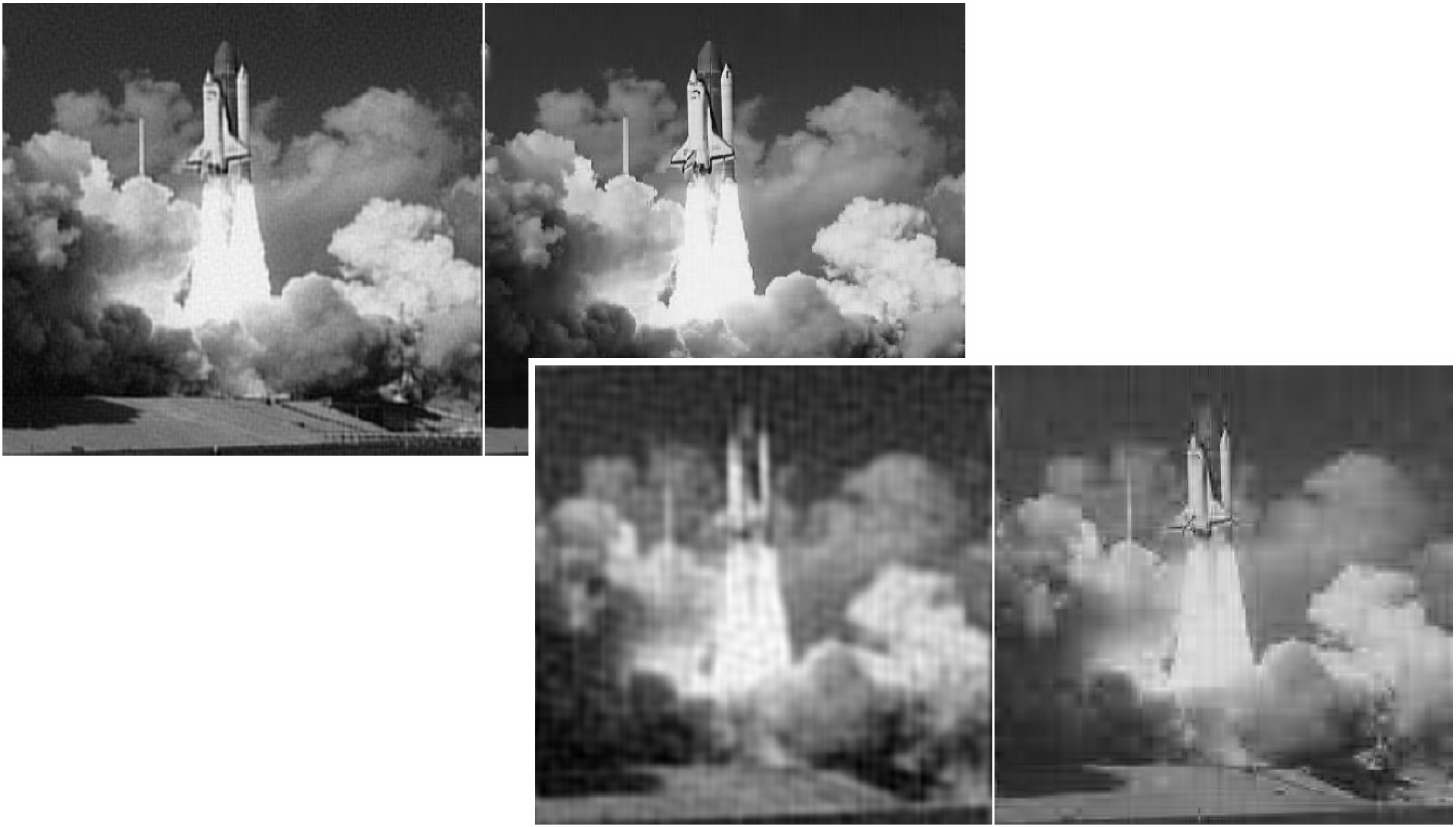


Compression (1)

- Wavelet calculation
- Find small coefficients and discard them
- Store only remaining coefficients
- Lossy compression

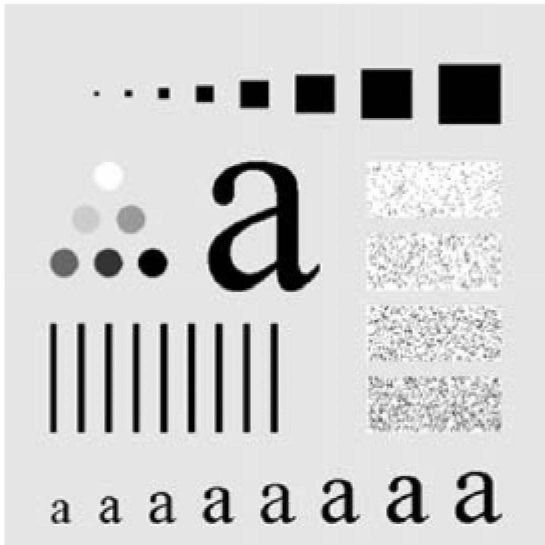
- Good compression with a fast convergence speed of the wavelet and good decay of the coefficients

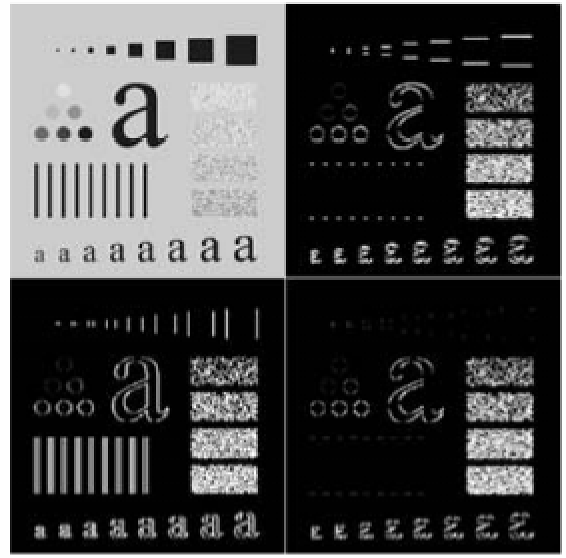
Compression (2)



Edge detection

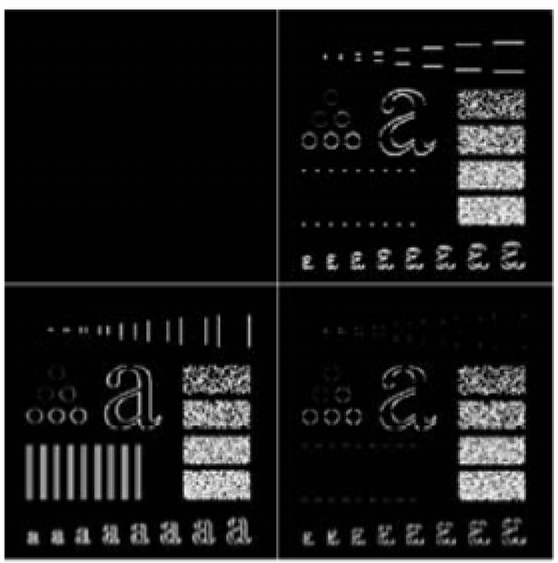
original





decomposition

approximation
is set to zero



reconstruction

References

- [1] Wavelets and Subband Coding, Martin Vetterli, Jelena Kovacevic, ISBN 0-13-097080-8
- [2] Wavelets – praktische Aspekte, Markus Grabner, VO2006
- [3] AK Computergrafik Bildverarbeitung und Mustererkennung WS 2006/07

Series Expansion with Wavelets

Thank you for you attention!

Advanced Signal Processing 2 2007
Teichtmeister Georg
Reinisch Bernhard