Series Expansion with Wavelets

Advanced Signal Processing 2 - 2007
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Introduction

- Series expansion
- Fourier Series: Either periodic or bandlimited signals
  - Timedomain: No frequency information
  - Fourierdomain: No time information
- Is there something between?
Contents

- Basics of signal representation
- Wavelets
  - Haar wavelet
  - Multiresolution analysis
  - Construction of the Sinc - Wavelet
- Wavelets derived from iterated filter banks
  - Haar case, Sinc case, general construction
- Wavelet series and its properties
- Practical outlook (image processing)
Recap of Series expansion

• Signals are points in a Vectorspace
• Time-Domain: Basis functions are infinite short impulses
• Signals can be projected onto other basis functions

\[ f(t) = \sum_{k=-\infty}^{\infty} \langle \varphi_k(u), f(u) \rangle \varphi_k(t) \]

\[ \langle \varphi_k(u), f(u) \rangle = \int_{-\infty}^{\infty} \varphi_k^*(u)f(u)du \]
Possible Basis Functions

- Fourier series
  - periodic
  \[ f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{j(2\pi k t)/T} \]

- Fourier transform
  - bandlimited

- STFT
  - Infinite set of Fourier Transforms

- Piecewise Fourier Series

- Wavelets

- Piecewise Fourier Series
Short Time Fourier Transform

- Window Signal
- Compute the Fourier Transform
- Shift window and repeat
  ⇒ Spectrogram, Periodogram
Time and Frequency Resolution

- Window has Energy distribution in both: Frequency ($\sigma_\omega$) and Time ($\sigma_n$).
- Uncertainty principle:

$$\sigma_\omega \sigma_n \geq \frac{1}{2}$$

- Optimality is only reached by Gaussian window
STFT T/F-Resolution

- Constant over Time and Frequency
Piecewise Fourierseries

- Fourier Series with non-overlapping rectangular windows in time and periodic expansion
- Why?
  - Overlapping windows are redundant information
  - Good Time Resolution
  - Representation of arbitrary functions
- Bad Frequency Resolution
- Errors at boundaries
Desired Features of Basis Functions

- Simple characterization
- Localization Properties in Time and Frequency
- Invariance under certain operations
- Smoothness properties
- Moment properties
Haar - Expansion

• Simplest Wavelet Expansion

• Scaled and shifted Wavelets:

\[ \phi_{m,n}(t) = 2^{-m/2} \phi(2^{-m} t - n) \]

• m … Scale
• n … Timeshift
Dyadic Tiling

- Resolution depends on Frequency now
- \( \phi_{m,n}(t) = 2^{-m/2} \phi(2^{-m} t - n) \)
Orthonormal Basis for $L_2$?

- Two wavelets on the same Scale have no common support
- Shorter wavelet always averages to zero
- Shifting so that jump matches, is not possible
Proof: Definitions

• Consider functions which are constant on
  \[ [n2^{-m_0}, (n+1)2^{-m_0}] \]
• and have finite support on
  \[ [-2^{m_1}, 2^{m_1}] \]
• Can approximate \( L_2 \) arbitrarily well
• We call it \( f^{(-m_0)}(t) \)
Proof: Scaling function

- The scaling function

\[
\varphi_{-m_0,n}(t) = \begin{cases} 
2^m & n2^{-m_0} \leq t < (n+1)2^{-m_0} \\
0 & \text{otherwise}
\end{cases}
\]

- Approximating the piecewise constant function

\[
f^{(-m_0)}(t) = \sum_{n=-N}^{N-1} f_n^{(-m_0)} \varphi_{-m_0,n}
\]

\[
N = 2^{m_0 + m_1}
\]

\[
f_n^{(-m_0)} = 2^{\frac{-m_0}{2}} f^{(-m_0)}(n2^{-m_0})
\]
Proof: Illustration

![Diagrams](image)

**Figure 4.3** Haar wavelet decomposition of a piecewise continuous function. Here, $m_0 = 0$ and $m_1 = 3$. (a) Original function $f^{(0)}$. (b) Average function $f^{(1)}$. (c) Difference $d^{(1)}$ between (a) and (b). (d) Average function $f^{(2)}$. (e) Difference $d^{(2)}$. (f) Average function $f^{(3)}$. 

Wavelet construction
Proof: Keystep

- Examination of two adjacent Intervals

\[ [2n2^{-m_0}, (2n+1)2^{-m_0}) \text{ and } (2n+1)2^{-m_0}, (2n+2)2^{-m_0} ) \]

- Now \( f^{(-m_0)}(t) \) can be expressed as

\[ f^{(-m_0)}_{2n} \phi_{-m_0,2n}(t) + f^{(-m_0)}_{2n+1} \phi_{-m_0,2n+1}(t) \]

- For \( m_0=0, n=1 \), this means

\[ [2,3) \text{ and } [3,4) \]

\[ f^{(0)}_{2} \phi_{0,2}(t) + f^{(0)}_{3} \phi_{0,3}(t) = 2\phi_{0,2}(t) + 3\phi_{0,3}(t) \]
Proof: Average and Difference

The function $f^{(-m_0)}(t)$ can also be expressed as the average

$$f^{(-m_0)}_{2n} + f^{(-m_0)}_{2n+1} \frac{\sqrt{2} \phi_{-m_0+1,n}(t)}{2}$$

and the difference

$$f^{(-m_0)}_{2n} - f^{(-m_0)}_{2n+1} \frac{\sqrt{2} \phi_{-m_0+1,n}(t)}{2}$$

over two intervals
Proof: Coefficients

- With

\[
 f_n^{(-m_0+1)} = \frac{1}{\sqrt{2}} \left( f_{2n}^{(-m_0)} + f_{2n+1}^{(-m_0)} \right)
\]

\[
 d_n^{(-m_0+1)} = \frac{1}{\sqrt{2}} \left( f_{2n}^{(-m_0)} - f_{2n+1}^{(-m_0)} \right)
\]

- We get

\[
 f_{2n}^{(-m_0)} \varphi_{-m_0,2n}(t) + f_{2n+1}^{(-m_0)} \varphi_{-m_0,2n+1}(t) =
\]

\[
 f_n^{(-m_0+1)} \varphi_{-m_0+1,n}(t) + d_n^{(-m_0+1)} \phi_{-m_0+1,n}(t)
\]
Proof: Finalization

- Applying all the things we can write

\[ f^{(-m_0)}(t) = f^{(-m_0+1)}(t) + d^{(-m_0+1)}(t) = \]

\[ = \sum_{n=-N/2}^{N/2-1} f^{(-m_0+1)}_n \phi_{m_0+1,n}(t) + \sum_{n=-N/2}^{N/2-1} d^{(-m_0+1)}_n \phi_{m_0+1,n}(t) \]

- Repeating the Average/Difference scheme for higher scales leads to

\[ f^{(-m_0)}(t) = f^{(m_1)}(t) + \sum_{m=-m_0+1}^{m_1} \sum_{n=-2^{m_1-m}}^{2^{m_1-m}-1} d^{(m)}_n \phi_{m,n}(t) = \]

\[ = \sum_{m=-m_0+1}^{m_1+M} \sum_{n=-2^{m_1-m}}^{2^{m_1-m}-1} d^{(m)}_n \phi_{m,n}(t) + \epsilon_M \]
Proof: Illustration

Figure 4.3 Haar wavelet decomposition of a piecewise continuous function. Here, $m_0 = 0$ and $m_1 = 3$. (a) Original function $f^{(0)}$. (b) Average function $f^{(1)}$. (c) Difference $d^{(1)}$ between (a) and (b). (d) Average function $f^{(2)}$. (e) Difference $d^{(2)}$. (f) Average function $f^{(3)}$. 
Multiresolution

- Successive approximation
- Coarse approximation + added details
- Coarse and detail subspace are orthogonal
- Leads to self-similar Wavelets in Scale
- Useful for applications
Axiomatic Definition (1)

• Sequence of embedded closed subspaces

\[ \ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots \]

• Upward Completeness

\[ \bigcup_{m \in \mathbb{Z}} V_m = L_2(R) \]

• Downward Completeness

\[ \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \]
Axiomatic Definition (2)

- Scale Invariance
  \[ f(t) \in V_m \iff f(2^m t) \in V_0 \]
- Shift Invariance
  \[ f(t) \in V_0 \Rightarrow f(t-n) \in V_n \quad \forall n \in \mathbb{Z} \]
- Existance of a orthonormal Basis
  – Non-orthogonal Basis can be orthogonalized
Orthogonal Complements

- \( V_m \) is a subspace of \( V_{m-1} \)
- We define \( W_m \) the orthogonal subset of \( V_m \) in \( V_{m-1} \)
- \( V_{m-1} = V_m \oplus W_m \)
- \( V_m \) is the space of the scaling functions
- \( W_m \) the space of the wavelets
- By repeating we get

\[
L_2(R) = \bigoplus_{m \in \mathbb{Z}} W_M
\]
Constructing the Sinc Wavelet

- Now the scaling functions will be the space of bandlimited functions.
- $V_0$ is bandlimited between $[-\pi, \pi]$, $V_{-1}$ between $[-2\pi, 2\pi]$.
- $W_0$ the functions bandlimited to $[-2\pi, -\pi]$ combined with $[\pi, 2\pi]$.
- $V_{-1} = V_0 \oplus W_0$.
Scaling function

- The scaling function is given by

\[ \varphi(t) = \frac{\sin \pi t}{\pi t} \]
Representation of $\phi$

- $V_0$ belongs to $V_{-1}$
- $\phi(t)$ can be represented by basis functions of $V_{-1}$
  \[
  \phi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)
  \]
  \[
  \|g_0[n]\| = 1; \quad g_0[n] = \sqrt{2} \langle \varphi(2t-n), \varphi(t) \rangle
  \]
- Without proof
  \[
  g_1[n] = (-1)^n g_0[-n+1]
  \]
  \[
  \phi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_1[n] \varphi(2t-n)
  \]
Construction Kernel

- $g_0$ is given by

$$g_0[n] = \frac{1}{\sqrt{2}} \frac{\sin(\pi n/2)}{\pi n/2}$$

$$G_0(e^{j\omega}) = \begin{cases} -\sqrt{2}e^{-j\omega} & -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

- And finally the wavelet

$$\phi(t) = \frac{\sin(\pi t / 2)}{\pi t / 2} \cos(3\pi t / 2)$$
Sinc Wavelet: Illustration

Figure 4.6 Scaling function and the wavelet in the sinc case. (a) Scaling function \( \varphi(t) \). (b) Fourier transform magnitude \( |\Phi(\omega)| \). (c) Wavelet \( \psi(t) \). (d) Fourier transform magnitude \( |\Psi(\omega)| \).
Iterated filter banks

- Until now we constructed wavelets by scaling and shifting of orthonormal function families
  - Based on multiresolution analysis

- Different approach by filter banks
  - Iteration leads to a wavelet
  - Key properties
    - regularity
    - degree of regularity
Haar case

- Low- and Highpass

\[ g_0 = \begin{bmatrix} \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \end{bmatrix}; \quad g_1 = \begin{bmatrix} \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \end{bmatrix} \]

- Iterate the filter bank on the lowpass channel

- Multirate signal processing results
Haar case

- size-8 discrete Haar transform

\[
g_0^{(i)}[n] = \begin{cases} 
2^{-i/2} & n = 0, \ldots, 2^i - 1, \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
g_1^{(i)}[n] = \begin{cases} 
2^{-i/2} & n = 0, \ldots, 2^i - 1 - 1, \\
-2^{-i/2} & n = 2^i - 1, \ldots, 2^i - 1, \\
0 & \text{otherwise}, 
\end{cases}
\]

- Number of coefficients grow exponentially

- Continuous time function

\[
\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n] \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i},
\]

\[
\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n] \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i}.
\]

- Length bounded, piecewise constant
Sinc case (1)

- Impulse responses (low and highpass filter)
  \[ g_0[n] = \frac{1}{\sqrt{2}} \frac{\sin(\pi / 2n)}{\pi / 2n} ; \quad g_1[n] = (-1)^n g_0[-n+1] \]

- Fourier transform
  \[
  G_0(e^{j\omega}) = \begin{cases} 
  \sqrt{2} & -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2}, \\
  0 & \text{otherwise}, 
  \end{cases} \quad G_1(e^{j\omega}) = \begin{cases} 
  -\sqrt{2}e^{-j\omega} & \omega \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi], \\
  0 & \text{otherwise}, 
  \end{cases}
  \]

- Now consider the iterated filter bank
  - Upsampling filter impulse
  - Emulate the Haar construction with \( g_0[n] \), \( g_1[n] \)
  - And define a scaling function
Sinc case (2)

Fourier transform of $\varphi^{(i)}(t)$

$$\Phi^{(i)}(\omega) = 2^{-i/2} G_0^{(i)}(e^{-j\omega/2^i}) e^{-j\omega/2^{i+1}} \frac{\sin(\omega / 2^{i+1})}{\omega / 2^{i+1}}$$

where:

$$G_0^{(i)}(e^{j\omega}) = G_0(e^{j\omega}) G_0(e^{j2\omega})... G_0(e^{j2^{i-1}\omega})$$

short:

$$M_0(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega})$$

we can rewrite:

$$\Phi^{(i)}(\omega) = \left[ \prod_{k=1}^{i} M_0(\frac{\omega}{2^k}) \right] e^{-j\omega/2^{i+1}} \frac{\sin(\omega / 2^{i+1})}{\omega / 2^{i+1}}$$

- For further analysis: important part is in the brackets
- This product is $2^i 2\pi$ periodic -> in the end it’s only a perfect lowpass (sinc scaling function)
Sinc case (3)

- Cumbersome way
- But we have gained a more general construction
- The key is the infinite product
  - Does this product converge and to what
  - Converge to what kind of scaling function
Iterated filter banks cont. (1)

- General construction
  - Two channel orthogonal filter bank
  - $g_0[n], g_1[n]$ are low- and highpass filter
  - Iterate on the branch of the lowpass filter and process this to infinity
  - Express the two filters after $i$-steps
  - Multirate conclusions
    - „Filtering with $G_i(z)$ followed by upsampling by 2 is equivalent to upsampling by 2 followed by filtering with $G_i(z^2)$“
Iterated filter banks cont. (2)

\[ G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k}), \]
\[ G_1^{(i)}(z) = G_1(z^{2^{i-1}}) \prod_{k=0}^{i-2} G_0(z^{2^k}), \quad i = 1, 2, \ldots \]

\[ \varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \quad n/2^i \leq t < \frac{n+1}{2^i}, \]
\[ \psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n], \quad n/2^i \leq t < \frac{n+1}{2^i}. \]

- Discrete time iterated filters combined with the continuous time functions
- Normalization and rescaling
- Graphical function
  - piecewise constant
  - halving the interval
Iterated filter banks cont. (3)

- Fourier domain act as above
- In the iteration scheme we are interesting in convergence

\[
\varphi(t) = \lim_{i \to \infty} \varphi^{(i)}(t), \quad \Phi(\omega) = \lim_{i \to \infty} \Phi^{(i)}(\omega) = \prod_{k=1}^{\infty} M_0 \left( \frac{\omega}{2^k} \right),
\]

\[
\psi(t) = \lim_{i \to \infty} \psi^{(i)}(t), \quad \Psi(\omega) = \lim_{i \to \infty} \Psi^{(i)}(\omega) = M_1 \left( \frac{\omega}{2} \right) \prod_{k=2}^{\infty} M_0 \left( \frac{\omega}{2^k} \right),
\]

- This will lead us to regularity discussion
Regularity

- The existence of the limit are critical conditions
  - Limits exist if $g_0[n]$ are regular
  - Regular filter leads through iteration to a scaling function with some degree of smoothness (regularity)
  - But not only convergence is sufficient we need also $L_2$ convergence to build orthonormal bases
  - A lot of sufficient conditions, different approaches
Wavelet series and properties

- Enumeration of some general properties of basis functions

\[ f(t) = \sum_{m,n \in \mathbb{Z}} F[m, n] \psi_{m,n}(t) \]

\[ F[m, n] = \langle \psi_{m,n}(t), f(t) \rangle = \int_{-\infty}^{+\infty} \psi_{m,n}(t), f(t) dt \]

- Wavelet
  - Linearity, Shift, Dyadic sampling and time frequency tiling, Scaling, Localization, decay properties
Linearity

suppose operator T

\[ T[f(t)] = F[m, n] = \left\langle \psi_{m,n}(t), f(t) \right\rangle \]

then for any \( a, b \in \mathbb{R} \)

\[ T[a \, f(t) + b \, g(t)] = aT(f(t)) + bT(g(t)) \]

- The wavelet series is linear. The proof follows from the linearity of the inner product
Shift

• For Fourier transform
  – pair: f(t), F(ω) … f(t-tau), e^{-jωtau} F(ω)

• Now for the wavelet series

\[
F'[m, n] = \int_{-\infty}^{+\infty} \psi_{m,n}(t) f(t - \tau) dt
\]

\[
F'[m, n] = \int_{-\infty}^{+\infty} 2^{-m/2} \psi(2^{-m} t - n + 2^{-m} \tau) f(t) dt
\]

\[
2^{-m} \tau \in Z \text{ or } \tau = 2^m k, k \in Z
\]

\[
f(t - 2^m k) \leftrightarrow F[m', n - 2^{m-m'} k], m' < m
\]
Scaling

• For Fourier transform
  – pair: \( f(t), F(\omega) \) … \( f(at), 1/a*F(\omega/a) \)

\[
F'[m, n] = \int_{-\infty}^{+\infty} \psi_{m,n}(t) f(at) \, dt
\]

\[
F'[m, n] = 1/a \int_{-\infty}^{+\infty} 2^{-m/2} \psi\left( \frac{2^{-m} t}{a} - n \right) f(t) \, dt
\]

\( a = 2^{-k}, k \in \mathbb{Z} \)

\( f(2^{-k} t) \leftrightarrow 2^k F[m - k, n] \)
Dyadic sampling and time frequency tiling

• It is important to locate the basis functions in the time-frequency plane

• sampling in time, at scale $m$, with period $2^m$
  $$\psi_{m,n}(t) = \psi_{m,0}(t - 2^m n)$$

• The frequency is the inverse of scale, we find if the wavelet is centered around $\omega_0$ then:
  $$\Psi_{m,n}(\omega)$$ is centered around $\omega_0 / 2^m$

• This leads to dyadic sampling of time frequency plane
Dyadic sampling

- The dots indicate the center of the wavelets
- The scale axis is logarithmic
Time localization (1)

- Suppose we are interested in the signal around \( t = t_0 \)
- Which values of \( F[m,n] \) carry information about signal \( f(t) \) at \( t_0 \Rightarrow f(t_0) \)
- Suppose wavelet \( \psi(t) \) is supported on the interval \([-n_1, n_2]\)
  - \( \psi_{m,0}(t) \) is supported on \([-n_1 2^m, n_2 2^m]\)
  - \( \psi_{m,n}(t) \) is supported on \([(-n_1+n)2^m, (n_2+n)2^m]\)
Time localization (2)

- At scale $m$, wavelet coefficients with index $n$ satisfy
  \[ (-n_1 + n)2^m \leq t_0 \leq (n_2 + n)2^m \]
  can be rewritten
  \[ 2^{-m}t_0 - n_2 \leq n \leq 2^{-m}t_0 - n_1 \]
Frequency localization (1)

• Suppose now in localization, but now in frequency domain

\[ \psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m} t - n) \]

the Fourier transform is

\[ 2^{m/2} \Psi(2^m \omega) e^{-j2^m n \omega} \]

\[ F[m,n] = \int_{-\infty}^{+\infty} \psi_{m,n}(t)f(t)dt \]

\[ F[m,n] = \frac{1}{2\pi} 2^{m/2} \int_{-\infty}^{+\infty} F(\omega) \Psi(2^m \omega) e^{j2^m n \omega} d\omega \]
Frequency localization (2)

- Suppose that the wavelet vanishes in the Fourier domain outside the region $[\omega_{\text{min}}, \omega_{\text{max}}]$.
- At specific scale $m$, the support of $\Psi_{m,n}(\omega)$ will be $[\omega_{\text{min}}/2^m, \omega_{\text{max}}/2^m]$.
- Therefore, a frequency component $\omega_0$ influences at scale $m$:

$$\frac{\omega_{\text{min}}}{2^m} \leq \omega_0 \leq \frac{\omega_{\text{max}}}{2^m}$$

rewrite

$$\log_2 \left( \frac{\omega_{\text{min}}}{\omega_0} \right) \leq m \leq \log_2 \left( \frac{\omega_{\text{max}}}{\omega_0} \right)$$
Decay properties

- Fourier series can be used to characterize the regularity of a signal (decay of the transform coefficients)
  - Global regularity
- The wavelet transform can be used in a similar way
  - Local regularity
Multidimensional wavelets

- A way to build multidimensional wavelets is to use tensor products of their one-dimensional counterparts
  - This will lead to different „mother“ wavelets
  - Scale changes are now represented in matrices
  - offers diagonal scaling but is also more restricted
Practical aspects

- Wavelets in matlab

- Images
  - Image compression
  - Edge detection
  - De-noising
Matlab

- Matlab wavelet toolbox
  - Command line
    - Help wavelet
  - Gui tool (wavemenu)
    - 1D wavelets analysis
    - 2D wavelets analysis
    - De-noising
    - Image Fusion
    - Compression
Example 1D wavelet transform

- Discrete/continuous wavelet transform
- `coefs = cwt(S, SCALES, 'wname')`
Compression (1)

- Wavelet calculation
- Find small coefficients and discard them
- Store only remaining coefficients
- Lossy compression

- Good compression with a fast convergence speed of the wavelet and good decay of the coefficients
Compression (2)
Edge detection

original

decomposition

approximation

is set to zero

reconstruction

Wavelet construction
References

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Series Expansion with Wavelets

Thank you for your attention!

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