

Series Expansion with Wavelets

Advanced Signal Processing 2 – 2007

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1 Introduction

The topic of this report is Series Expansion with Wavelets. Wavelets are a family of basis functions for signals, either continuous or discrete in time. One of the most important properties of wavelets is the finite localization in time and frequency, as contrary to time representation (no localization in frequency) and frequency representations (no localization in time).

At the beginning the basics of signal representation will be covered, some examples will be given and their properties concerning time and frequency resolution will be discussed. Based on this discussion, some desired properties of basis functions for signals will be derived. The basic idea of wavelets will then be discussed by using the Haar – Wavelet (the first and probably also the simplest Wavelet) as an example. This will be done, by proofing, that the Haar – Wavelet is actually a basis for the space of functions which are squared summable (L_2 – space). In this step, the basic idea of wavelets, namely coarsely approximating the signal and then adding details are presented. This is called the Multiresolution concept, which properties will be highlighted, because of its importance to the Wavelet analysis.

The Haar – Wavelet itself is one boundary case for the Wavelet analysis, where as the other boundary is the Sinc – Wavelet. The Sinc - Wavelet is also shortly introduced and its basic construction steps are shown.

2 Signal representation

Signals are points in a vector space, which has infinite dimensions. As known, one needs basis functions which span the vector space to project the signal on. The choice of this basis functions and their properties will be the main topic of this report. To recap, a signal $f(t)$ is given by

$$f(t) = \sum_{k=-\infty}^{\infty} \langle \varphi_k(u), f(u) \rangle \varphi_k(t)$$
$$\langle \varphi_k(u), f(u) \rangle = \int_{-\infty}^{\infty} \varphi_k^*(u) f(u) du$$

φ_k are the basis functions on which the signal is projected. Now a few possibilities of basis functions and their properties will be discussed. Especially the influence of the basis functions on the localization in time and frequency. To clarify, we are talking about how well a coefficient of the signal representation is localized in both time and frequency or, in other words, what it tells us about the frequency and time properties of the signal.

2.1 Time domain representation

The time domain representation is the representation one obtains when measuring a signal. In this representation the basis functions are infinite short impulses:

$$\varphi_k(t) = \delta(t - kdt)$$
$$\varphi_k[n] = \delta(n - k)$$

As can be easily seen, the basis functions are precisely localized in time, but don't tell anything about the frequency (because you need to look at how the signal changes to say something about the frequency).

2.2 *Fourierseries and –transform*

This section covers the different types of Fourier expansion will be covered. First the Fourier Transform will be discussed. The Fourier Transform is one of the most used tools in frequency analysis.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Here the composition for a time discrete signal is shown. The main drawback of this method of representing signals the band limitation of the signals.

The more intuitive version of frequency analysis is the Fourier series. In this case the signal doesn't need to be band limited, but it has to be periodic. The basis functions in this case are infinite long sine and cosine waves.

$$f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{j(2\pi k t)/T}$$

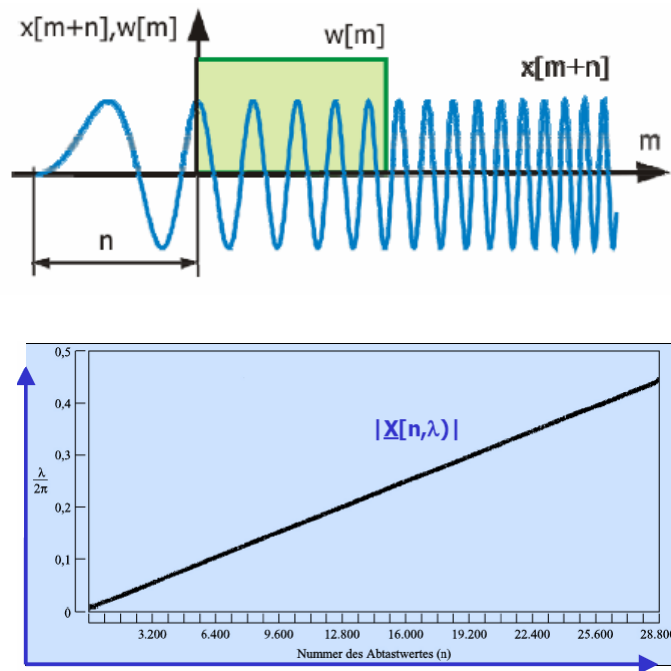
This equation shows the composition of a signal, when the coefficients of a Fourier series are given, and uses the complex notation.

To summarize: There are two methods for frequency analysis. But each has its own drawbacks. Whereas the Fourier transform can handle signals with finite support, the signals have to be band limited. On the other hand the Fourier series can represent signals with arbitrary frequencies, but the signals need to be periodic, which means they need to have infinite support.

As can easily seen for the Fourier series expansion, the localization in time is not existent (because the basis functions are defined from $t=-\infty$ bist $t=+\infty$), but the localization in frequency is infinitely sharp.

2.3 Short time fourier transform

Up to now we had representations of signals which are either perfectly localized in time (but infinitely bad in frequency), or perfectly localized in frequency (but infinitely bad in time). Often something in between is needed. One common problem is the detection of changes in the spectrum. When thinking of a change, normally the words “before” and “after” come to our minds, which implies some sort of time resolution. So to detect changes in the spectrum we need a spectrum with time resolution, e.g. the spectrum of the last second of the time signal. The short time fourier transform (STFT) is one tool, which can provide such information. When applying the STFT, the time signal will be windowed, and a standard Fourier Transform will be computed over the windowed signal. To get STFT for the whole signal, which results in a periodogram, the window is shifted over the time signal.



In the top figure a sine sweep ($x[n]$) and the window ($w[m]$) are shown, in the bottom figure the resulting periodogram. On the x-axis is the time index n and the y-axis indicates the frequency at that time.

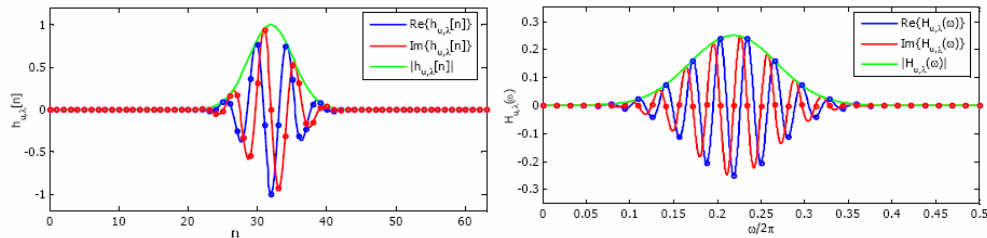
2.3.1 Time and frequency resolution

The analysis of the time and frequency resolution of the STFT is not that easy as in the previous cases. As said above, and as also seen in the periodogram, the STFT is actually a function of time *and* frequency. So the question about the accuracy of the STFT in time and frequency arises. The most important factor, which influences these accuracies, is the shape of the window function. If the window is wider, one can compute a more accurate Fourier transform, if the window is narrower, the time localization is much better. From this intuitive approach it is easily seen, that one can not optimize both frequency and time resolution. Actually there exists an equivalent to the Heisenberg uncertainty principle known from quantum mechanics. When taking a window, one can compute its Fourier transform and furthermore its distribution moments in the time and frequency domain. For the discrete

time/frequency case the product of the standard deviations has to be equal to or greater than a half.

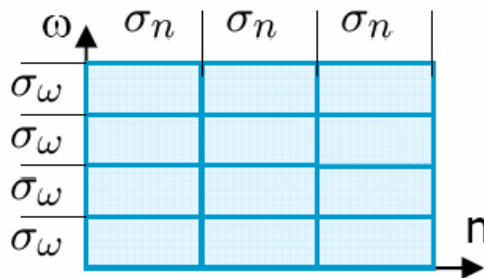
$$\sigma_{\omega}\sigma_n \geq \frac{1}{2}$$

Where σ_{ω} is the standard deviation of the window in frequency domain, and σ_n is the standard



deviation of the window in time domain. The value of $\frac{1}{2}$ is just reached, if one uses a Gaussian window.

One other important property of the time/frequency resolution of the STFT is its equality over all frequencies and time as seen below.



When talking about the time resolution, it is of course desired that it is invariant with respect to the time parameter. On the other hand it is not always desired that the frequency resolution is independent of the frequency. The adaptation of the frequency resolution (and hence the time resolution too) to frequency is one major advantage of the Wavelet expansion.

2.4 Piecewise Fourier series

Another possibility to achieve finite time and frequency resolution, would be to window the time domain signal with a rectangle, continue it periodically and compute the Fourier series expansion of this new signal.

Contrary to the STFT, which computes the Fourier transform of overlapping windows, this approach doesn't compute redundant information. Furthermore one can represent arbitrary signals from the L_2 space, because the Fourier series expansion enables one to represent band unlimited signals, and the way of constructing the signal, which is going to be used as input for the Fourier series expansion, guarantees that there is always a periodic signal, even if the original signal isn't periodic.

On the other hand there also some major drawbacks: Because of the non-overlapping windows, the result of this expansion is determined by the length and the location of the windows. E.g. if one wants to represent a pure sine with this method, the result changes if the length of the window is not a multiply of the sine period and the window is placed on different locations of the original signal. This happens because of the discontinuities at the window borders.

2.5 *Desired Features of basis functions*

Up to now a few ways of representing signals were introduced. All of them have their advantages and drawbacks. Now the question arises, which properties are desirable and which are not?

- Simple characterization
If one wants to use signal representations for practical use, it is of course useful if it is simple to use
- Localization properties in both, time and frequency domain
On the one hand localization properties of the basis functions should be well defined, on the other hand one should be able to choose what resolution to use, and of course they should be as good as possible.
- Invariance under certain operations
The result of a transformation should not change if some operations are applied on the original signal. This property can be relaxed a little bit in a way, so that if you apply an operation on the original signal, the same operation is visible on the output of the transformation. E.g. if a time shift is applied on the original signal, one could not expect that the output of a STFT is the same as before the time shift, but the result is also shifted in time
- Smoothness properties
Sometimes it is useful, that basis functions can be derived or are continuous.
- Moment properties
Another thing which could be useful sometimes are certain moment properties. Especially zero moments make computations much easier.

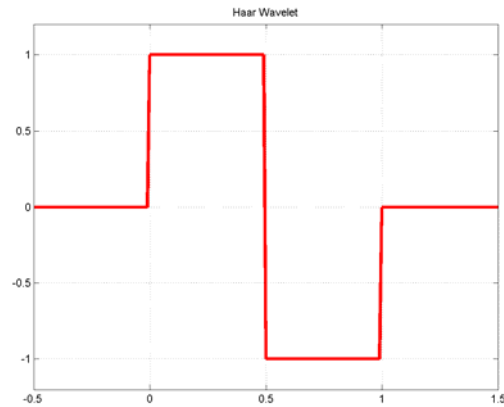
As can be seen, some of those properties conflict with each other, so that there is no optimal basis for function spaces. However, if an application is known, one can choose which properties are needed and then derive or just select a useful set of basis functions.

3 **Introduction to Wavelets**

In this chapter the basic ideas of Wavelets will be presented and discussed. First the idea of Wavelets is intuitively introduced using the Haar – Wavelet, after that a proof is given that the Haar – Wavelets are actually basis functions for the L_2 – space, the concept of multi resolution analysis is highlighted and finally the Sinc – Wavelet is introduced to show the differences to the Haar – Wavelet.

3.1 The Haar - Wavelet

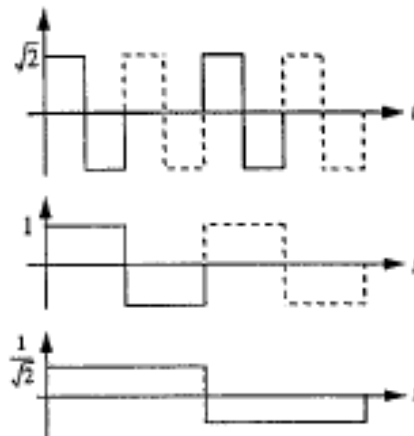
The Haar - Wavelet is one of the simplest Wavelet functions. The mother - wavelet is +1 between 0 and 0.5, and -1 between 0.5 and 1. The wavelet is shown in the picture below.



To represent the whole space of squared sumabel functions, this mother - wavelet is scaled and shifted.

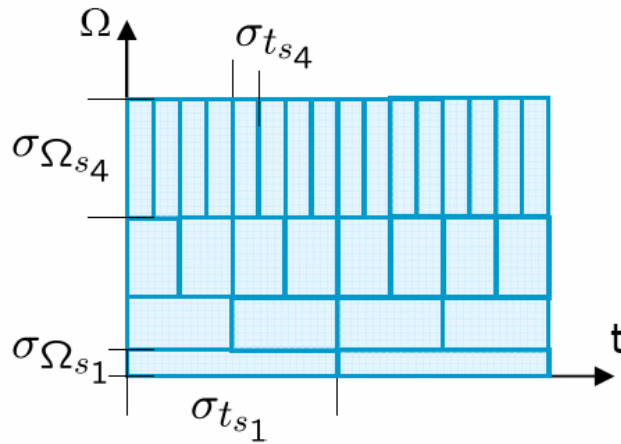
$$\phi_{m,n}(t) = 2^{-m/2} \phi(2^{-m}t - n)$$

In the above equation $\phi_{m,n}$ is the scaled and shifted version of the mother wavelet (ϕ). As can be easily seen, m indicates the scale and n the shift.



In this figure we see the mother wavelet and its shifted versions in the middle row, $\phi_{-1,n}$ in the top row and $\phi_{1,n}$ in the bottom row.

What was now achieved by using this way of constructing a basis for the L_2 - space? One of the most important things is, that the frequency and time resolution is now depending on the frequency.



As can be seen in the figure, the higher the frequency, the better the time resolution but the frequency resolution suffers. One thing has to be highlighted: The area of this boxes (which is the product of the standard deviations) always stays the same! What are the advantages of using such a scheme (dyadic tiling of the time/frequency plane)? The two main reasons are: First, things which have a high frequency tend to happen fast (so good time localization is needed), second, low frequencies can be better discriminated, the relative minimal difference between two frequencies which can be differentiated stays the same over all frequencies.

3.1.1 Ortho-normality of the Haar – Wavelet

One major requirement for basis is their ortho-normality. Of course one can construct non-ortho-normal basis too, but ortho-normal basis have some nice properties, which makes it easy to use them. A set of basis functions $\phi_{m,n}$ is ortho-normal, iff it satisfies following constraint:

$$\langle \phi_{m,n}(t), \phi_{m',n'}(t) \rangle = \delta[m-m']\delta[n-n']$$

That means, only the inner product of a wavelet at the same scale and with the same shift is equal to 1, every inner product of wavelets with different scales and/or shifts have to be zero. Are Haar – Wavelets fulfilling this constraint? Yes, they do.



When staying the same scale, wavelets are always shifted, so that they don't overlap. Therefore the projection wavelets in the same scale (identical m) but with different shifts (different n) is always zero. But what happens with different scales? In the figure above it is seen that the longer wavelet is constant over the support of the shorter one, and therefore the inner product is the average of a Haar – Wavelet, which is of course equal to zero. Just one thing has to be added: Because of the construction rule, it is not possible, that the jumps of the shorter and longer wavelet are at the same position, so this reasoning is valid for the general case.

3.1.2 Basis for L_2 – Space

Above it was shown that the Haar – Wavelets are actually ortho-normal to each other, but do they also form a basis for the space of functions in L_2 ? Luckily they are. One intuitive

explanation is based on the composition of a signal. A signal is given by the sum over all basis functions weighted with some coefficients. Using the construction rule for Haar – Wavelets, one could make m arbitrary large (that means the wavelet gets narrower and narrower) and therefore one can represent all functions of L_2 using infinite such infinite small Wavelets. However, there is a more formal proof, which will be presented now. This proof also presents the concepts of multi resolution analysis of signals.

To start the proof, lets consider functions which are constant on an interval $[n2^{-m_0}, (n+1)2^{-m_0}]$ and have finite support on the interval $[-2^{m_1}, 2^{m_1}]$. It is obvious, that if choosing both m_0 and m_1 large enough, every L_2 function can be approximated arbitrarily well. Such piecewise constant functions are called $f^{(-m_0)}(t)$ from now on.

This piecewise constant functions can be represented by a sum of indicator functions $\varphi_{-m_0,n}(t)$:

$$\varphi_{-m_0,n}(t) = \begin{cases} 2^{\frac{m_0}{2}} & n2^{-m_0} \leq t < (n+1)2^{-m_0} \\ 0 & otherwise \end{cases}$$

The height of the indicator functions is scaled, so that they have unit norm. This function is called the scaling function. As said above the piecewise constant function $f^{(-m_0)}(t)$ can be written as a sum of indicator functions:

$$f^{(-m_0)}(t) = \sum_{n=-N}^{N-1} f_n^{(-m_0)} \varphi_{-m_0,n}$$

$$N = 2^{m_0+m_1}$$

$$f_n^{(-m_0)} = 2^{\frac{-m_0}{2}} f^{(-m_0)}(n2^{-m_0})$$

Where N is just a stating how wide the support and how detailed the piecewise constant function is, and $f_n^{(-m_0)}$ is the coefficient for the particular scaled and shifted scaling function.

The key step in this proof is the relation between two adjacent intervals:

$[2n2^{-m_0}, (2n+1)2^{-m_0}]$ and $[(2n+1)2^{-m_0}, (2n+2)2^{-m_0}]$. The piecewise constant function $f^{(-m_0)}(t)$ over these two intervals is:

$$f_{2n}^{(-m_0)} \varphi_{-m_0,2n}(t) + f_{2n+1}^{(-m_0)} \varphi_{-m_0,2n+1}(t)$$

Just the sum of two versions of the scaling function with their respective coefficients. But there is another way to describe the function over these two intervals. One can compute the average of the two intervals and then add the difference between the average and the original function. To compute the average, the scaling function with $m = -m_0 + 1$ is used (that scaling function is as wide as both intervals together), to compute the difference to the original function, the Wavelet of scale $m = -m_0 + 1$ is used. Remember, this special wavelet has actually two parts which directly correspond to the old intervals. The mathematical expression of the average

$$\frac{f_{2n}^{(-m_0)} + f_{2n+1}^{(-m_0)}}{2} \sqrt{2} \varphi_{-m_0+1,n}(t)$$

and the difference using the wavelet function $\phi_{-m_0+1,n}(t)$:

$$\frac{f_{2n}^{(-m_0)} - f_{2n+1}^{(-m_0)}}{2} \sqrt{2} \phi_{-m_0+1,n}(t)$$

To ease the derivation a little bit, we introduce coefficients for the Scaling function and the wavelet, $f_n^{(-m_0+1)}$ and $d_n^{(-m_0+1)}$ respectively:

$$f_n^{(-m_0+1)} = \frac{1}{\sqrt{2}} (f_{2n}^{(-m_0)} + f_{2n+1}^{(-m_0)})$$

$$d_n^{(-m_0+1)} = \frac{1}{\sqrt{2}} (f_{2n}^{(-m_0)} - f_{2n+1}^{(-m_0)})$$

This results in following expression for our original two intervals:

$$f_n^{(-m_0+1)} \phi_{-m_0+1,n}(t) + d_n^{(-m_0+1)} \phi_{-m_0+1,n}(t)$$

Just the average over the two intervals plus the difference. To represent the whole function $f^{(-m_0)}(t)$, the explained mechanism is just applied to all adjacent pairs of the whole function. After doing this, following representation is achieved:

$$\begin{aligned} f^{(-m_0)}(t) &= f^{(-m_0+1)}(t) + d^{(-m_0+1)}(t) = \\ &= \sum_{n=-N/2}^{N/2-1} f_n^{(-m_0+1)} \phi_{-m_0+1,n}(t) + \sum_{n=-N/2}^{N/2-1} d_n^{(-m_0+1)} \phi_{-m_0+1,n}(t) \end{aligned}$$

The piecewise constant function $f^{(-m_0)}(t)$ is represented as the piecewise constant function with longer constant intervals (actually twice as long), plus the error which is made when computing the average. The second equation is just the more detailed representations with all the scaling functions and wavelets with their respective coefficients. To state again, the original function is decomposed into “average” (coarse) and “difference” (details).

Now the final step will be applied: Decompose $f^{(-m_0+1)}(t) = f^{(-m_0+2)}(t) + d^{(-m_0+2)}(t)$, that means the old average in a new average and details and do this repeatedly. Finally

$$f^{(-m_0)}(t) = f^{(m_1)}(t) + \sum_{m=-m_0+1}^{m_1} \sum_{n=-2^{m_1-m}}^{2^{m_1-m}-1} d_n^{(m)} \phi_{m,n}(t) = \sum_{m=-m_0+1}^{m_1+M} \sum_{n=-2^{m_1-m}}^{2^{m_1-m}-1} d_n^{(m)} \phi_{m,n}(t) + \varepsilon_M$$

is obtained. Where $f^{(m_1)}(t)$ is a residual average function. However, it can be shown that the norm of this residual decreases when it is decomposed further. However this decomposition doesn't contribute to the basic idea of the proof, so it will be skipped. The residual is expressed as ε_M in the right representation. As the residual error can be made arbitrary small, and m_0 and m_1 can be made arbitrary large, this shows that the Haar – Wavelet is sufficient to represent all functions from L_2 . The scaling function is *not* needed.

The key of the proof and also the key idea of wavelets, is the decomposition of a signal in a coarse approximation and the details which get lost during that coarse approximation. This is

also the basic property of the multi resolution analysis, which will be covered in the next chapter.

$$\bigcup_{m \in \mathbb{Z}} V_m = L_2(\mathbb{R})$$

4 Multi resolution Analysis

The multi resolution concept directly follows from the construction of wavelets. Repeatedly applying the decomposition of a fine signal in a coarse approximation and the difference between the fine signal and coarse approximation leads to a representation which just needs successive details to represent L_2 signals.

It will be shown, that the coarse approximation and the details to add are orthogonal subspace of the space of the fine signal.

4.1 Axiomatic Definition

- Enclosed Subspaces
 $\dots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \dots$
- Upward Completeness
 $\bigcup_{m \in \mathbb{Z}} V_m = L_2(\mathbb{R})$
- Downward Completeness
 $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$
- Scale Invariance
 $f(t) \in V_m \Leftrightarrow f(2^m t) \in V_0$
- Shift Invariance
 $f(t) \in V_0 \Rightarrow f(t - n) \in V_n \quad \forall n \in \mathbb{Z}$
- Existence of an ortho-normal basis

To make things a bit clearer, this axioms will be explained starting from the top. When talking about enclosed subspaces as defined, it is important to note that signals which can be represented by V_0 can also be represented by V_{-1} as V_0 is subset of V_{-1} .

Upward Completeness means, that all Subspaces combined result in the L_2 space, which means such a multi resolution approach can actually represent those signals. Downward completeness means, that there is no error left.

Scale Invariance is kind of clear. If you scale a function which can be represented by V_m by 2^m it can then be represented by V_0 . It is also the same with shift invariance: Shifted versions of a function are still represented by the same subspace.

Finally this subspaces have to be spanned somehow, so there has to be a basis for them. As one can normalize basis, it is not needed that those basis are ortho-normal from the beginning.

4.2 Orthonormal Complements

In the last section the axioms which are needed for multi resolution analysis were defined and discussed. But how does this fit together with wavelets and the proof which was discussed above?

As said above, V_m is a subspace of V_{m-1} . That means, if you subtract V_m of V_{m-1} , there has to be something left. Actually this is the ortho-normal complement of V_m in V_{m-1} . It will be called W_m . That means, if V_m and W_m are combined, the result is V_{m-1} .

$$V_{m-1} = V_m \oplus W_m$$

Now the spaces V_m are said to be the spaces of the scaling functions and W_m are the spaces of the wavelets. By repeating the process of splitting up V_{m-1} into its subspace V_m and its orthogonal complement W_m for all m , we get

$$L_2(R) = \bigoplus_{m \in Z} W_M$$

This is equivalent with the result we got above, when proving that the Haar Wavelet is actually able to represent all functions of the L_2 space.

5 Constructing the Sinc – Wavelet

While discussing the basic properties of wavelets, the Haar – Wavelet was used as an example. One thing which was not covered in the discussion about wavelets up to now was their frequency and time resolution, except that the wavelet construction introduces the dyadic tiling of the time/frequency plane. The actual size of the box was not discussed. Thinking about the Haar – Wavelet, which is basically build up from two rectangular signals, it is intuitively seen that it will have a good localization in time but a bad one in frequency. The opposite of the spectrum, good localization in frequency but a bad one in time, is reached when using the Sinc – wavelet. The Haar – Wavelet uses a scaling function which has limited support in time, the Sinc – wavelet uses a scaling function which has limited support in frequency, hence is band limited. As the name already suggests, this is the Sinc – function.

$$\varphi(t) = \frac{\sin \pi t}{\pi t}$$

The space V_0 will include the functions which are band limited to $[-\pi, +\pi]$, V_{-1} the functions which are band limited to $[-2\pi, +2\pi]$. Using the knowledge gained in the multi resolution section, W_0 will then be then be the functions which are band limited to $[-2\pi, -\pi]$ and $[\pi, +2\pi]$. As can be easily seen, combining V_0 and W_0 results in V_{-1} . So what is the wavelet function?

As already discussed, V_0 is a subspace of V_{-1} , so functions in V_0 can be represented by functions in V_{-1} . As $\varphi(t)$ is in V_0 , it can be represented by functions in V_{-1} :

$$\varphi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t - n)$$

$$\|g_0[n]\| = 1; g_0[n] = \sqrt{2} \langle \varphi(2t - n), \varphi(t) \rangle$$

The function g_0 and its fourier transform G_0 is the characteristic function of a multiresolution analysis and the key step in construction wavelets, when starting with a scaling function. Without proof it is shown, how to construct g_1 and then the mother wavelet.

$$g_1[n] = (-1)^n g_0[-n + 1]$$

$$\phi(t) = \sqrt{2} \sum_{n \in Z} g_1[n] \varphi(2t - n)$$

For the sinc – case these functions are given by:

$$g_0[n] = \frac{1}{\sqrt{2}} \frac{\sin(\pi n / 2)}{\pi n / 2}$$

$$G_0(e^{j\omega}) = \begin{cases} -\sqrt{2}e^{-j\omega} & -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

And finally the sinc – wavelet

$$\phi(t) = \frac{\sin(\pi t / 2)}{\pi t / 2} \cos(3\pi t / 2)$$

In the figure below, the scaling function and the wavelet are shown.

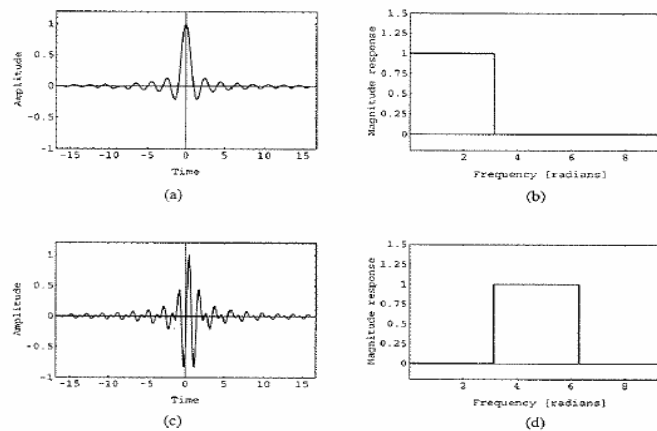


Figure 4.6 Scaling function and the wavelet in the sinc case. (a) Scaling function $\phi(t)$. (b) Fourier transform magnitude $|\Phi(\omega)|$. (c) Wavelet $\psi(t)$. (d) Fourier transform magnitude $|\Psi(\omega)|$.

6 Iterated filter banks

Until now we have constructed wavelets by using orthonormal families of functions by scaling and shifting. This continuous time approach is directly based on the axioms of multiresolution analysis.

From now on we will derive wavelets by using discrete time filters. These filters can be iterated under certain conditions and will also lead to continuous time wavelets. We will have a recap of the Haar and Sinc case which can be seen as limits from iterated filter banks. This construction can be generalized. The key property of such discrete time filters is the regularity condition. Discrete filters can be called regular if they converge to a scaling function with some degree of regularity (piecewise smooth, continuous, or derivable).

6.1 Haar case

We start now with the discrete Haar filters. We know that the lowpass averages two neighboring samples and the highpass builds the difference of them. The filter bank for this is given by the basis functions g_0 and g_1 . Now we start to iterate the filter bank on the lowpass channel (see Figure 1).

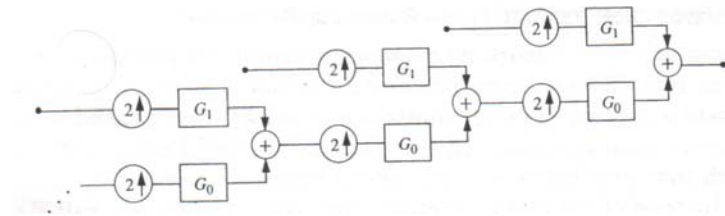


Figure 1: Iterated filter bank

We now will derive from this construction an equivalent filter bank by using results from multirate signal processing. Filtering by g_0 followed by upsampling by two is the same to upsampling by two, followed by filtering by g'_0 (= upsampled version of g_0). By doing so we can transform the iterated filter bank into one equivalent to a filter which is shown in Figure 2.

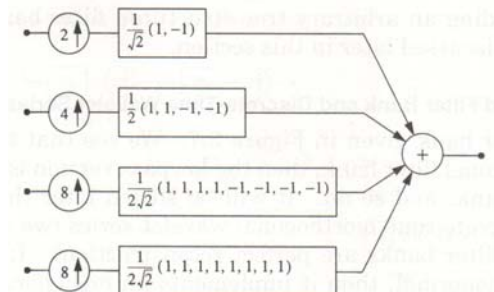


Figure 2: Transformed filter bank

This is a size 8 discrete Haar transform of successive blocks of 8 samples. By iteration of the lowpass and highpass channels we will come to two filters (Figure 3):

$$g_0^{(i)}[n] = \begin{cases} 2^{-i/2} & n = 0, \dots, 2^i - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_1^{(i)}[n] = \begin{cases} 2^{-i/2} & n = 0, \dots, 2^{i-1} - 1, \\ -2^{-i/2} & n = 2^{i-1}, \dots, 2^i - 1, \\ 0 & \text{otherwise,} \end{cases}$$

Figure 3: Iterated filter equations

From the filter equations we can see, that g_0 is a lowpass filter and g_1 is a bandpass filter. We also can see a interesting property. If we increase the iterations depth the length of the filter also increases exponentially and on the other side the coefficients go to zero.

If we now define the continuous time function with g_0 and g_1 in this way:

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n], \quad n/2^i \leq t < \frac{n+1}{2^i},$$

$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n], \quad n/2^i \leq t < \frac{n+1}{2^i}.$$

Figure 4: Continuous time functions

These functions are piecewise constant and their lengths remain bounded.

6.2 Sinc case

As a counterpart for the Haar case we now have a look at the sinc case. In this case the filters in the filter bank are ideal low- and highpass filters. The impulse response for the lowpass filter is:

$$g_0[n] = \frac{1}{\sqrt{2}} \frac{\sin(\pi/2n)}{\pi/2n}$$

The perfect high pass filter can be constructed by modulating $(-1)^n$ and shifting by one. For the impulse response we get with use of the lowpass filter:

$$g_1[n] = (-1)^n g_0[-n+1]$$

Then we transform both impulse responses and we will get:

$$G_0(e^{j\omega}) = \begin{cases} \sqrt{2} & -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2}, \\ 0 & \text{otherwise,} \end{cases} \quad G_1(e^{j\omega}) = \begin{cases} -\sqrt{2}e^{-j\omega} & \omega \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi], \\ 0 & \text{otherwise,} \end{cases}$$

Figure 5: Fourier transform of $g_0[n]$

Figure 6: Figure 5: Fourier transform of $g_1[n]$

We now will do the same as in the Haar case. We consider the iterated filter bank with ideal filters. For instance upsampling the filter responses by two will lead to a filter with a smaller bandwidth (and so on for further iterations). In a related way the transfer function of the highpass filter can be constructed and leads to a bandpass filter.

Now we have our filter we can emulate the Haar construction for the low- and highpass impulse responses $g_0[n]$ and $g_1[n]$ which are the iterated filters.

Then we can define a $\varphi(t)$ as we have seen in Figure 4. In the sinc case we are interested in the Fourier transform of this scaling function and we will obtain

$$\Phi^{(i)}(\omega) = 2^{-i/2} G_0^{(i)}(e^{-j\omega/2^i}) e^{-j\omega/2^{i+1}} \frac{\sin(\omega/2^{i+1})}{\omega/2^{i+1}}$$

where the iterated transfer function can be rewritten as

$$G_0^{(i)}(e^{j\omega}) = G_0(e^{j\omega}) G_0(e^{j2\omega}) \dots G_0(e^{j2^{i-1}\omega}) \quad M_0(\omega) = \frac{1}{\sqrt{2}} G_0(e^{j\omega})$$

and we will end up with a compact version of the Fourier transform:

$$\Phi^{(i)}(\omega) = \left[\prod_{k=1}^i M_0\left(\frac{\omega}{2^k}\right) \right] e^{-j\omega/2^{i+1}} \frac{\sin(\omega/2^{i+1})}{\omega/2^{i+1}}$$

So now let's have a look on the different parts of this formula. The right part (exponential and sin-function) is just a phase factor and interpolation function and if we let the number of iterations i grow this part becomes nearly one for any finite frequency ω .

So let's have a look on the first part in the brackets. The product is a $2^i 2\pi$ periodic function between $-\pi$ and $+\pi$. So if we let i grow, we will end up with a perfect lowpass filter from $-\pi$ and $+\pi$ which is the sinc scaling function. For the wavelet function we can do it similarly and we will end up with known results.

This seems to be a very cumbersome way to derive results we already have known. But we have gained a more general construction and we have seen the key for this analysis is the product in the brackets. For regularity analysis we have to investigate this infinite product.

6.3 Iterated filter banks cont.

Now we will use the derivation of the Haar and sinc wavelets using iterated filter banks to make a more general construction for wavelet bases. Once again we start with our two filters $g_0[n]$ and $g_1[n]$, which are low- and highpass filters. For the iterated filter bank we will iterate the on the lowpass channel branch (to infinity). As we have done this in the two constructions above we will now reformulate our particular filters by their iterated representations (by using multirate identities). So we will end up with following transfer functions for our filters:

$$G_0^{(i)}(z) = \prod_{k=0}^{i-1} G_0(z^{2^k}),$$

$$G_1^{(i)}(z) = G_1(z^{2^{i-1}}) \prod_{k=0}^{i-2} G_0(z^{2^k}), \quad i = 1, 2, \dots$$

Then we will associate the iterated filters with the time functions to obtain the scaling function and the wavelets as follows:

$$\varphi^{(i)}(t) = 2^{i/2} g_0^{(i)}[n] \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i},$$

$$\psi^{(i)}(t) = 2^{i/2} g_1^{(i)}[n] \quad \frac{n}{2^i} \leq t < \frac{n+1}{2^i}.$$

These functions had to be rescaled and normalized for regularity reasons. In Figure 7 we can see four iterations of a length 4 filter. We can see the scaling function φ is a piecewise constant approximation and in every iteration we halve the interval.

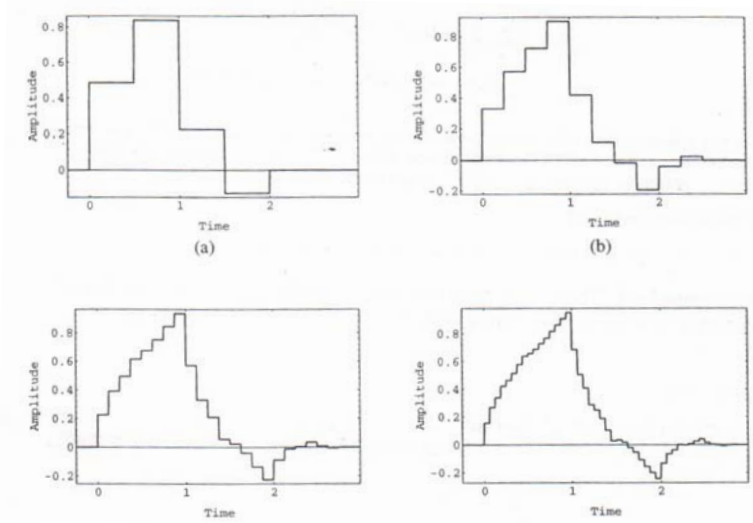


Figure 7: Iterated filter equations

For the Fourier domain we can do the same as we have done in the sinc case. For the iterated scheme we have to investigate the limits of the scaling and wavelet functions.

$$\varphi(t) = \lim_{i \rightarrow \infty} \varphi^{(i)}(t),$$

$$\psi(t) = \lim_{i \rightarrow \infty} \psi^{(i)}(t).$$

As we have figured out in the sinc case we only focus our interest on the term in the brackets (because the second part converges to one). For the Fourier domain the limit will be

$$\Phi(\omega) = \lim_{i \rightarrow \infty} \Phi^{(i)}(\omega) = \prod_{k=1}^{\infty} M_0\left(\frac{\omega}{2^k}\right),$$

$$\Psi(\omega) = \lim_{i \rightarrow \infty} \Psi^{(i)}(\omega) = M_1\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} M_0\left(\frac{\omega}{2^k}\right),$$

And it turns out that these two functions are really scaling functions which can be obtained as limits of discrete time iterated filters.

The existences of these limits are very critical conditions. We can easily say that these limits exist if the filters are regular. That means that regular filters leads, through iteration to a scaling function. The key for the regularity analysis is the behavior of the infinite product. There are convergence questions and if convergence exist we have also answer in what sense the convergence exist and which properties can be fulfilled. In this report we will not dig deeply into these discussions. There are several approaches to investigate this behavior which gives us sufficient conditions. Some of them are also necessary but also to restrictive. In general we can say that this part is still an open topic for research.

7 Wavelet series and properties

The purpose of this section is to have a look on the wavelet construction and onto the properties. The following will be an enumeration of some general properties. First we review and define the problem as follows

$$f(t) = \sum_{m,n \in \mathbb{Z}} F[m,n] \psi_{m,n}(t)$$

$$F[m,n] = \langle \psi_{m,n}(t), f(t) \rangle = \int_{-\infty}^{+\infty} \psi_{m,n}(t) f(t) dt$$

where ψ is the mother wavelet and $f(t)$ is any function in the $L_2(\mathbb{R})$ space.

7.1 Linearity

If we suppose an operator T which is defined as

$$T[f(t)] = F[m, n] = \langle \psi_{m, n}(t), f(t) \rangle$$

then for any $a, b \in \mathbb{R}$

$$T[af(t) + bg(t)] = aT(f(t)) + bT(g(t))$$

Therefore we can say that the wavelet operator is linear (so far unproven but it can be followed from the linearity of the inner product).

7.2 Shift

First we have a look on the shift property of the Fourier transform where a function $f(t)$ transforms into $F(\omega)$ and a time shift of the function $f(t-\tau)$ transforms into $\exp(-j\omega\tau) * F(\omega)$. What happens now with the wavelet series. The function and its coefficients are written as $f(t)$ and $F[m, n]$. If now $f(t)$ is shifted by τ , we investigate the function $f(t-\tau)$:

$$F[m, n] = \int_{-\infty}^{+\infty} \psi_{m, n}(t) f(t - \tau) dt$$

$$F[m, n] = \int_{-\infty}^{+\infty} 2^{-m/2} \psi(2^{-m}t - n + 2^{-m}\tau) f(t) dt$$

If we now rewrite the shift as a multiple of $2^{-m}\tau$ then we can figure out following transform property:

$$2^{-m}\tau \in \mathbb{Z} \text{ or } \tau = 2^m k, k \in \mathbb{Z}$$

$$f(t - 2^m k) \leftrightarrow F[m', n - 2^{m-m'} k], m' < m$$

7.3 Scaling

Once again we first have a look on the scaling property of the rare Fourier transform which is given by a function $f(t)$ and its transform $F(\omega)$. If we now scale $f(t)$ with a parameter a so that $f(at)$ then the Fourier transform is $1/a * F(\omega/a)$.

For the wavelet expansion we can write

$$F[m, n] = \int_{-\infty}^{+\infty} \psi_{m, n}(t) f(at) dt$$

$$F[m, n] = 1/a \int_{-\infty}^{+\infty} 2^{-m/2} \psi\left(\frac{2^{-m}t}{a} - n\right) f(t) dt$$

Similarly as in the shift case we can rewrite the scaling factor as a power of two and we will come to following property

$$a = 2^{-k}, k \in \mathbb{Z}$$

$$f(2^{-k}t) \leftrightarrow 2^k F[m - k, n]$$

7.4 Dyadic sampling and time frequency tiling

When we have a look on the series expansion it is important to locate the basis functions in the time frequency plane. The sampling in time, at a specific scale m takes place with a period 2^m when we fulfill

$$\psi_{m,n}(t) = \psi_{m,0}(t - 2^m n)$$

On the other side frequency is the inverse of scale, we find when the wavelet is centered around ω_0 then $\Psi_{m,n}(\omega)$ is centered around $\omega_0/2^m$. This leads to dyadic sampling of the time frequency plane. In Figure 8 we can see such a time frequency plane where the scale axis is logarithmic and in relation to this we have a wavelet representation where we have a linear scale axis (Figure 9).

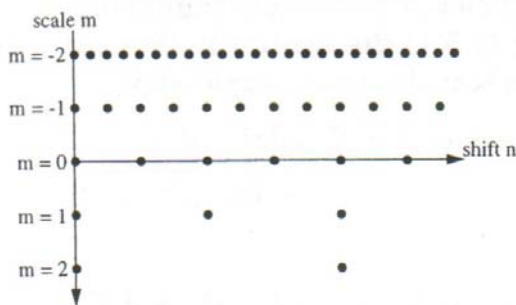


Figure 8: Dyadic sampling (logarithmic)

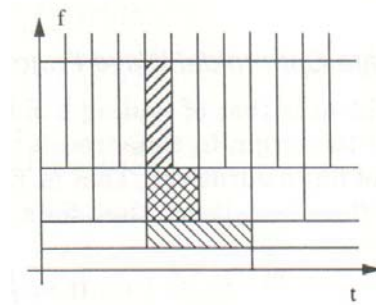


Figure 9: Dyadic sampling (linear)

7.5 Time localization

In general localization is the major advantage of the wavelet representations. For the time localization we suppose that we are interested in the signal around a specific time $t = t_0$. Now we want to know which values of $F[m,n]$ are related to the signal and which values of $F[m,n]$ carry information of the signal $f(t_0)$.

First suppose a wavelet which supports $\Psi(\omega)$ in the interval $[-n_1, n_2]$. Then $\Psi_{m,0}(\omega)$ is in the range of $[-n_1 2^m, n_2 2^m]$ and furthermore $\Psi_{m,n}(\omega)$ is supported $[(-n_1+n)2^m, (n_2+n)2^m]$. At a specific scale m , the wavelet coefficients with index n satisfy that

$$(-n_1 + n)2^m \leq t_0 \leq (n_2 + n)2^m$$

can be rewritten

$$2^{-m} t_0 - n_2 \leq n \leq 2^{-m} t_0 - n_1$$

In Figure 10 we can see which region carries information about the scale in the wavelet $F[m,n]$.

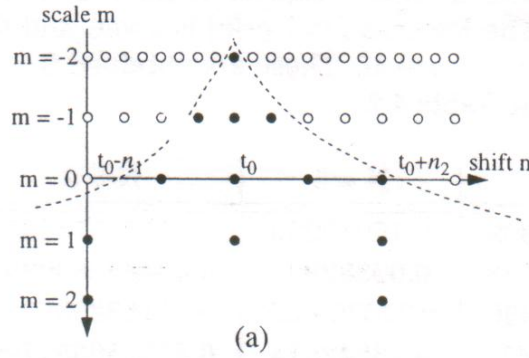


Figure 10: Time localization

7.6 Frequency localization

Similarly as we have done in the time localization we know interested which region in $F[m,n]$ carries some information about the frequency localization. First we have a look at the Fourier transform of $\psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m}t - n)$ which is $2^{m/2} \Psi(2^m \omega) e^{-j2^m n \omega}$. Then we can write

$F[m,n]$ as

$$F[m,n] = \int_{-\infty}^{+\infty} \psi_{m,n}(t) f(t) dt$$

$$F[m,n] = \frac{1}{2\pi} 2^{m/2} \int_{-\infty}^{+\infty} F(\omega) \Psi(2^m \omega) e^{j2^m n \omega} d\omega$$

To narrow the region of interest we say the wavelets vanishes in the Fourier domain outside a specific region $[\omega_{\min}, \omega_{\max}]$ (the coefficients need not to vanish but most of the energy should be in there). At a specific scale the support of $\Psi_{m,n}(\omega)$ will be $[\omega_{\min}/2^m, \omega_{\max}/2^m]$. So we can say that a frequency component around a specific frequency ω_0 influences the wavelet series at scale m if

$$\frac{\omega_{\min}}{2^m} \leq \omega_0 \leq \frac{\omega_{\max}}{2^m}$$

is satisfied and we can rewrite for the scale

$$\log_2 \left(\frac{\omega_{\min}}{\omega_0} \right) \leq m \leq \log_2 \left(\frac{\omega_{\max}}{\omega_0} \right)$$

In Figure 11 we can see which values will be influenced from a specific frequency.

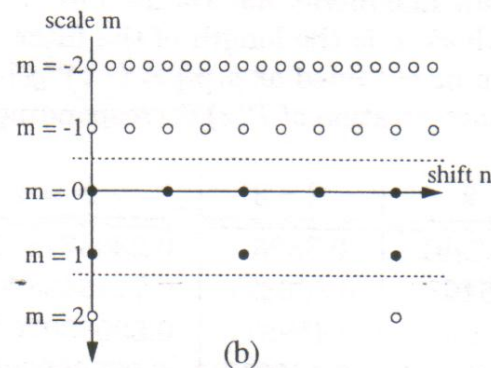


Figure 11: Frequency localization

7.7 Decay properties

The Fourier series can be used to characterize the regularity of a signal, which can be done by looking on the decay of the transform coefficients. The drawback of the Fourier analysis is that we only come down to a global regularity of the signal. This is another advantage of the wavelet transform. When we look here on the coefficients of the transform we can estimate a local regularity of the signal.

7.8 Multidimensional wavelets

For applications like image compression, we can extend the theories of one dimensional filter banks can be extended to multiple dimensions. A direct way to construct multidimensional wavelets is to use tensor products and their one dimensional counterparts. This will lead us to one scaling function with three different mother wavelets. The scale changes will now represented in a matrix which offers several advantages but also restrictions on the matrix. The link to filter banks can be easily found but the task is much harder (incomplete cascade structures).

8 Practical aspects

This section should give a short overview where to start in matlab and what possible applications are for instance in image processing.

In matlab there is whole toolbox which deals with wavelet implementations. For a short overview have a look on the “wavemenu”. There are several possibilities to use and different approaches. The only suggestion we can give here is to start at this point and work through topics which you are interested in.

9 References

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