Introduction to ST Codding

Graz University of Technology
Institute of Signal Processing and Speech Communication

Seminar report by
Yohannes Alemseged
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Outline of the Report

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1 Introduction

Space Time Codes(STC) were first introduced by Tarokh et al.from AT&T labs in 1998 as a novel means of providing transmit diversity for the multiple-antenna fading channel. Other temporal diversity techniques like temporal diversity, frequency diversity and receive antenna diversity were used to deal with multipath fading in multiple antenna wireless systems before the STC came into scene.

STC is a means /technique to address broad goals in maximizing link performance, maximize link throughput and minimize error.The supporting performance criteria to meet this goal could be signaling rate (in bps/Hz or bits per transmission), diversity gain (or diversity order, which is the slope of the error vs SNR curve), coding gain (from code design that increases effective SNR), and the array gain (from antenna combining that also increases effective SNR).

Improving error performance is possible by maximizing diversity whose upper bound is $M_T M_R$)in MIMO channels. A well designed STC codes can ensure this upperbound. Increasing coding gain also depends on the minimum distance of the code. Certain classes of ST codes (STTC) which are discussed in this report can provide coding gain as well as diversity gain.

2 Coding and interleaving architecture

We now discuss the general coding architecture for transmission over multiple antennas shown in Fig 1. A block of qK bits is input for temporal coding, interleaving and symbol mapping and in the process $q(N-K)$ parity bits are added and N symbols are output.

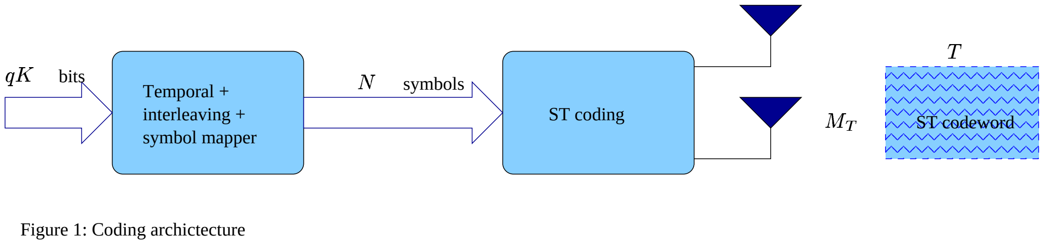


Figure 1: Coding Arc

Nsymbols are added to the ST coder that adds an additional $M_T T - N$ parity symbols and packs the resulting $M_T T$ symbols into an $M_T \times T$ frame of length T. The block/frame is then transmitted over T symbol periods and is noted as an ST codeword. Signaling data rate is qK/T bits/transmission and this should be should be with in channel capacity.

$$\frac{qK}{T} = q\left(\frac{qK}{qN}\right)\left(\frac{N}{T}\right) = qr_t r_s \tag{1}$$

r_t is temporal code rate of the outer encoder and r_s is the spacial code rate defined as the average number of independent symbols transmitted from the M_T antennas over T symbol periods.

The spacial code rate r_s varies between 1 to M_T. When all transmit antennas send one symbol per symbol period we get $r_s = 1$. In Spatial Multiplexing scheme we send $M_T$independent symbols per symbol period to get $r_s = M_T$.

Interleaving is used to spread burst errors that could occur due to fades across codewords to improve error correction performance. Interleaving is absolutely necessary in ST coding to exploit all available spatial diversity by mitigating any space selective fading across transmit antennas.

3 ST coding for frequency flat channels

3.1 Signal model

Consider a MIMO system with M_T and M_R and a code word $M_T \times T$ dimension. The code word can be denoted by $\mathbf{S} = [\mathbf{s}[1]\mathbf{s}[2] \dots \mathbf{s}[T]]$, where $\mathbf{s}[k] = [s_1[k] \dots s_{M_T}[k]]^T$, is the transmitted vector symbol over the kth symbol period.

$$\mathbf{y}[k] = \sqrt{\frac{E_s}{M_T}}\mathbf{Hs}[k] + \mathbf{n}[k], k = 1, 2, \dots, T, \tag{2}$$

$$\mathbf{Y} = \sqrt{\frac{E_s}{M_T}}\mathbf{HS} + \mathbf{N}, k = 1, 2, \dots, T, \tag{3}$$

where $\mathbf{Y} = [\mathbf{y}[1]\mathbf{y}[2] \dots \mathbf{y}[T]]$ and $\mathbf{N} = [\mathbf{n}[1]\mathbf{n}[2] \dots \mathbf{n}[T]]$
are matrices of size $M_R \times T$

For decoding a transmitted codeword, a receiver is assumed to use a Maximum Likelihood **ML** detection with perfect channel knowledge. The estimated codeword is:

$$\widehat{S} = \arg\min_s \|\mathbf{Y} - \mathbf{HS}\|_F^2$$

$$= \arg\min_s \|\mathbf{y}[k] - \sqrt{\frac{E_s}{M_T}}\mathbf{Hs}[k]\|_F^2 \tag{4}$$

Minimization is performed over all admissible codewords S

3.2 ST codeword design criteria

To extract the parameters for codeword design, first we define the pair wise error probability (PEP). Given that the receiver constructs a ML estimate of the transmitted codeword according to the equation in 4, the probability that the receiver mistakes the transmitted codeword $\mathbf{S}^{(i)}$for another codeword $\mathbf{S}^{(j)}$ is referred to us the Pair Wise Error Probability.

$$P(\mathbf{S}_{(i)} \rightarrow \mathbf{S}_{(j)}|\mathbf{H}) = Q\left(\sqrt{\frac{E_s\|\mathbf{H}(\mathbf{S}_{(i)} - \mathbf{S}_{(j)})\|_F^2}{2M_T N_o}}\right)$$

$$= Q\left(\sqrt{\frac{\rho\|\mathbf{HE}_{i,j}\|_F^2}{2M_T}}\right) \tag{5}$$

$\mathbf{E}_{i,j} = \mathbf{S}^{(i)} - \mathbf{S}^{(j)}$ is the $M_T \times T$ codeword difference matrix and $\rho = E_s/N_0$ is the **SNR**, applying Chernoff bound to the PER we get;

$$P(\mathbf{s}_{(i)} \rightarrow \mathbf{S}_{(j)}|\mathbf{H}) \leq e^{-\frac{\rho\|\mathbf{HE}_{i,j}\|_F^2}{4M_T}} \tag{6}$$

After some computational manipulation our PEP equation reduces to;

$$P(\mathbf{s}_{(i)} \rightarrow \mathbf{S}_{(j)}|\mathbf{H}) \leq \frac{1}{\left(\prod_{k=1}^{r(\mathbf{G}_{i,j})} \lambda_k(\mathbf{G}_{i,j})\right)^{M_R}} \left(\frac{\rho}{4M_T}\right)^{-r(\mathbf{G_{i,j}})\mathbf{M_R}} \tag{7}$$

where $\lambda_k(\mathbf{G}_{i,j})(k = 1, 2, 3, \ldots, r(\mathbf{G}_{i,j}))$ are non-zero eigenvalues of $\mathbf{G}_{i,j} = \mathbf{E}_{i,j}\mathbf{E}_{i,j}^H$. The above equation (7) leads to the two-well known criteria for ST codeword construction "rank criterion" and "determinant criterion".

3.2.1 Rank criterion

The rank criterion optimizes the spatial diversity extracted by an ST code. Referring equation (7), the ST code extracts $r(\mathbf{G_{i,j}})M_R$ order diversity. $r(\mathbf{G_{i,j}})$is the rank of $\mathbf{G_{i,j}}$.

To extract the full spatial diversity gain of $M_T M_R$, the code design should be such that $\mathbf{E}_{i,j}$ between any pair of codewords is full-rank ($r(\mathbf{G}_{i,j}) = \mathbf{M_T}$).

3.2.2 Determinant criterion

The determinant criterion optimizes the coding gain. Referring eq (7), the coding gain depends on the term:

$\left(\prod_{k=1}^{r(\mathbf{G}_{i,j})} \lambda_k(\mathbf{G}_{i,j}) \right)$

For high coding gain, this term should be maximum over all possible pairs of codeword matrices $\mathbf{S}^{(i)}$ and $\mathbf{S}^{(j)}$

3.3 ST diversity coding ($r_s \leq 1$)

Here two flavors of ST diversity codes are presented for discussion, STTC and STBC, which extract full diversity order ($M_T M_R$)with spatial rate $r_s \leq 1$

3.3.1 STTC

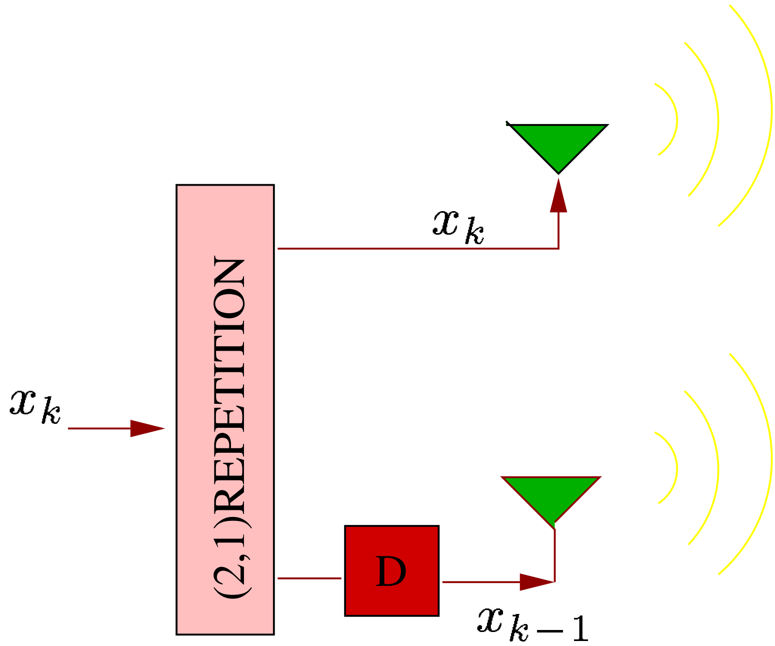
STTC are an extension of convolutional trellis codes to multiantenna systems. They map an arbitrary number of information symbols to antenna outputs according to a finite-state machine.

These codes can be designed to extract diversity gain and coding gain using the mentioned criteria in eq (7). Their ability to provide both kinds of gain makes them superior in performance over their STBC counterparts.

The simplest example of a STTC is the delay-diversity scheme depicted bellow, This transmitter can be modeled as a finite state machine, whose state at time $k$is k_{k-1}. The previous and current sequence of input symbols uniquely defines a trellis path. The number of trellis states is equal to the size of the input alphabet and the receiver can can implement ML sequence detection using Viterbi algorithm.

Fixing the number of input symbols to k, we can construct the following space-time block code A.

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_{K-1} & x_K & 0 \\ 0 & x_1 & x_2 & x_3 & \ldots & x_{K-1} & x_K \end{bmatrix} \tag{8}$$



Two zeros are inserted in the first and the last columns to initialize and terminate the trellis to a known state.

The code in the scene is a linear code and $\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}')$ between two distinct codewords reduces to $\mathbf{A}(\mathbf{e})$, whose rank is 2 and the delay diversity achieves full rank according to the rank criterion.

Another good example for improved delay diversity is a case of 8-PSK alphabet, $(A = e^{jl\pi/4})$.

$$\mathbf{A}(\mathbf{l}) = \begin{bmatrix} l_1 & l_2 & l_3 & \ldots & \\ 0 & 5l_1 & 5l_2 & 5l_3 & \ldots \end{bmatrix} \tag{9}$$

In the above representation the parity symbol index is 5l(modulo8). Identifying symbols by their integer lables l, the (2,1)reptition code in eq (8) can be written us $C = \{00, 15, 22, 37, 44, 51, 66, 73\}$ and the code for the 8-PSK alphabet can be written as $C = \{00, 15, 22, 37, 44, 51, 66, 73\}$, the Min equilidean distance for the former code is $d_{min} \approx 1.082$ and the latter code achieves a maximal minimum Euclidean distance of $d_{min} = 2$.

Thus using the code in eq (9) in place of the reption code in eq (8) in the delay-diversity transmitter structure, two antenna 8-state 8-PSK STTC is achieved

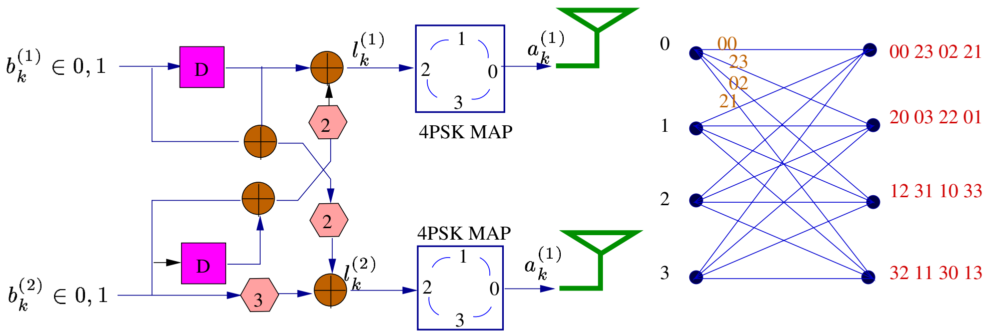


Fig 2 Trellis diagram for a 4-QAM, four state trellis code for M_T=2 with a rate of 2bps/Hz

Both the mentioned delay-diversity and the space-time trellis code above satisfy the rank criterion for full diversity achievement. However, the 8-PSK STTC shown, performs better because its coding gain is larger.

In fig(2) A space time trellis code for 4-PSK is shown, it is similar to the previous example except the fact that it is constrained to use a 4 PSK alphabet.

As indicated on the diagram, the mappers convert an integer $l \in 0, 1, 2, 3$ a 4-PSK symbol a according to $a = e^{(\frac{jl\pi}{2})}$.The number of nodes in the trellis diagram corresponds to the number of states. M_T entries in each constellation correspond to the symbols to be transmitted from the M_T antennas.There are four group of symbols for the four possible inputs (4-QAM constellation). Each group has two entries corresponding to the symbols to be output through the two transmit antennas

Decoding will be done using ML sequence estimation using Viterbi algorithm.

Important points to note

STTC are an effective means of capturing diversity. Increasing the number of states also increases the coding gain. However, the computational load for decoding an STTC increases exponentially with the number of states.

3.3.2 STBC

STBC are generalizations of Alamouti's scheme (two-antenna transmit diversity scheme)and we start discussion of the STBC by presenting the Alamouti's scheme.

Alamouti's scheme is a scheme which improves the signal quality at the receiver on one side of the link by simple processing across two transmit antennas at the opposite end. Bellow is the schemetic diagram of Alamouti's scheme.

Given that s_1 and s_2 to be transmitted, the Alamouti scheme transmits

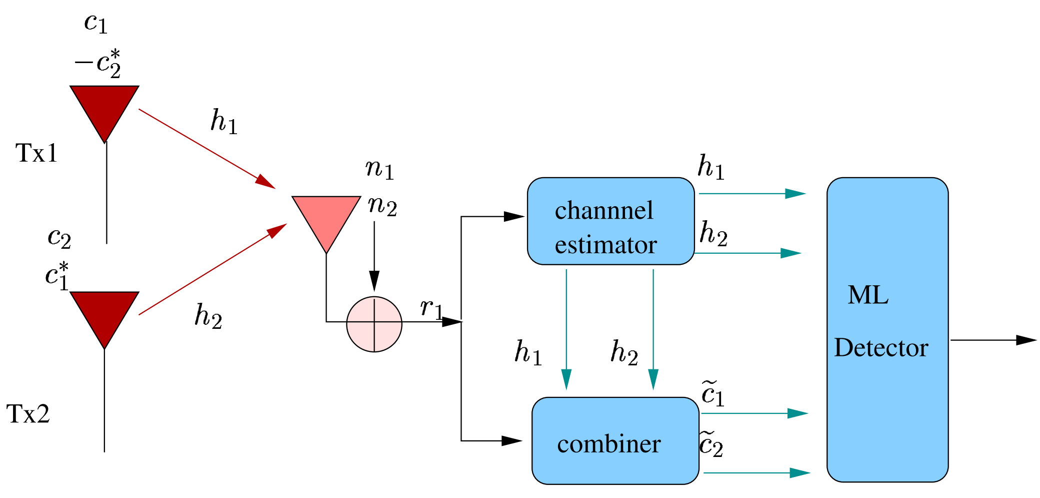


Fig3 Alamouti's Two Antenna Transmit Diversity Scheme

symbol s_1 and s_2 from antenna 1 and antenna 2 respectively in the first symbol period, followed by $-s_2^*$ and s_1^* from antenna 1 and antenna 2 respectively during the next symbol period.

The transmitted code word can be expressed as:

$$\mathbf{S} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} \tag{10}$$

The codeword difference matrix between any pair of codewords($S^{(i)}$ and $S^{(j)}$)

$$\mathbf{E}_{i,j} = \begin{bmatrix} e_1 & -e_2^* \\ e_2 & e_1^* \end{bmatrix} \tag{11}$$

$E_{i,j}$ is an orthogonal matrix with two non-zero eigenvalues (rank 2) of equal magnitude. The Alamouti scheme therefore delivers full $2M_R$ order diversity where M_R represents the number of receive antennas.

Alamouti's receiver output can be stated us:

$$y_i = \sqrt{\frac{E_s}{2}}\|\mathbf{H}\|_F^2 s_i + n, \qquad i = 1, 2, \tag{12}$$

$y_i$is the scalar processed received signal corresponding to transmitted symbol s_i and n_i is ZMCSCG noise with variance $\|H\|_F^2 N_0$

ST code construction for Alamouti type scheme to an arbitrary number of transmit antennas is possible. An example of an orthogonal design for $M_T = 4$ is shown bellow:

$$\mathbf{S} = \begin{bmatrix} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & s_4 & -s_3 \\ s_3 & -s_4 & s_1 & s_2 \\ s_4 & s_3 & -s_2 & s_1 \end{bmatrix} \tag{13}$$

Symbols s_1, s_2, s_3, s_4 are all drawn from a real constellation. The difference matrix between such type codewords $E_{i,j}$ is orthogonal matrix.The average PEP in the high SNR regime for an orthogonal STBC(OSTBC)from eq (7):

$$P(\mathbf{s}_{(i)} \to \mathbf{S}_{(j)}|\mathbf{H}) \le \left(\frac{M_T}{\|\mathbf{E}_{i,j}\|_F^2}\right)^{M_T M_R} \left(\frac{\rho}{4M_T}\right)^{-M_T M_R} \tag{14}$$

From the above equation we learn that OSTBC extract the full diversity gain of $M_T M_R$. In the case of complex constellations, an orthogonal design with spatial rate 1 doesn't exist for systems more than two transmit antennas, an Alamouti's scheme is the only existing code for complex constilation codewords.

Orthogonal designs for $r_s = \frac{1}{2}$ and $r_s \ge \frac{1}{2}$ do exist for systems with any number of transmit antennas.

OSTBC are attractive due to their low complexity implementation; simple linear processing at the receiver can decouple the vector detection problem into simpler scalar detection problems resulting in a simple input-output relations.

We can generalize their main feature as the provision of full diversity with very simple decoding as compared to the requirement by STTC. If coding gain is required, however, they have to be concatenated with an outer code.

3.4 Spatial multiplexing as a ST code $(\mathbf{r_s} = \mathbf{M_T})$:

So far the discussion has been with spatial rate $r_s \le 1$ and diversity order $M_T M_R$ where ofcourse there was one or less independent symbol transmitted per symbol period over the M_T antennas. In this part we discuss Spatial Multiplexing (SM) where we transmit more or less M_T independent symbols per symbol period.

In an uncoded SM scheme $r_t = 1$ and $r_s = M_T$ the signaling rate will be: qM_T bits/transmission

The receiver on the other side of the link treats each received signal vector as a codeword, and performs ML decoding on every vector symbol. The code word difference matrix$E_{i,j}$ is now an $M_T \times 1$ vector and the $\mathbf{E}_{i,j}\mathbf{E}_{i,j}^H$ is matrix with one rank and thus the average PEP is written as:

$$P(\mathbf{s}^{(i)} \to \mathbf{S}^{(j)}) \leq \frac{1}{\lambda(\mathbf{G}_{i,j})^{M_R}} \left(\frac{\rho}{4M_T}\right)^{-M_R} \tag{15}$$

SM with no coding may be considered as a ST code with spatial rate M_T with M_R order diversity

3.4.1 Horizontal Encoding (HE)

In HE the bit stream is first demultiplexed into M_T separate streams to undergo temporal coding, interleaving and symbol mapping before transmission. The spatial rate for this scheme will be then $r_s = M_T$ and the signaling rate will be qr_tM_Tbits/transmission.

The HE scheme, like the uncoded SM, can at most achieve M_R order diversity since any given symbol is transmitted from only one transmit antenna and received by M_R receive antennas (source of sub-optimality). Coding gain of the scheme depends on the strength of the temporal code and array gain of M_R is achievable.

3.4.2 Vertical Encoding (VE)

In VE the bit stream undergoes temporal coding, interleaving and symbol mapping before demultiplexing and transmission. With this scheme optimality is achievable since potentially each bit can be spread across all antennas.

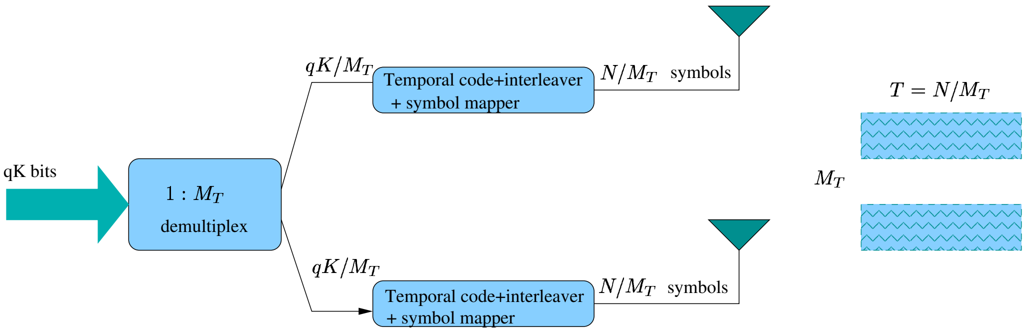


Figure 4 Horizontal Encoding (sub-optimal encoding technique that captures at most M_R order diversity

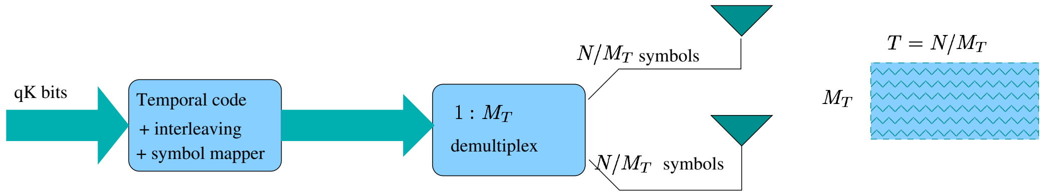


Figure 5 Vertical Encoding (allows spreading of information bits across all antennas, needs complex decoding tech)

The disadvantage of this scheme is, it requires joint decoding at the receiver and hence may be complex. The spatial rate and the signaling rate are the same as that of HE scheme. It can achieve a diversity order greater than M_R (bits are possibly spread over all antenna). Coding gain for this scheme depends on the temporal code design, and array gain of M_R is achievable

3.4.3 Diagonal Encoding (DE)

DE is a variation of the HE and VE schemes. Incoming data streams first undergoes HE encoding and split into frames/slots as shown in fig 6. The frames pass through a stream rotator so that the bit stream-antenna association is periodically cycled. For large codeword, the codeword from any one of the demultiplexed stream could be transmitted over all M_T antennas as in the special scheme called D-BLAST transmission technique.

In D-BLAST an initial wastage (no transmission) is required for optimal encoding, refer fig 7. Similar to the HE and VE, the spatial rate for DE is

M_T and the signaling rate is qr_tM_T bits/transmission.

In D-Blast type schemes M_TM_R diversity is possible if the stream rotation is optimal. In the this scheme also the coding gain depend on the temporal code gain and an array gain of M_R is achievable.

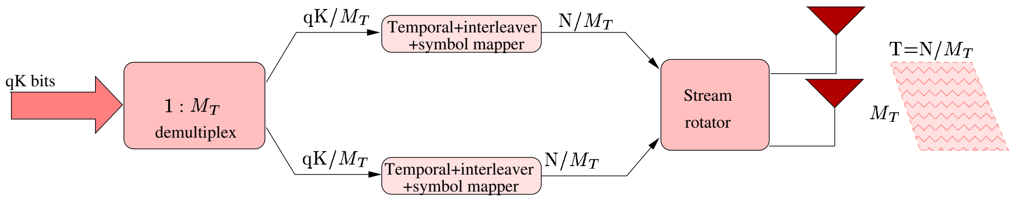


Fig6 Diagonal Encoding (HE) with stream rotation

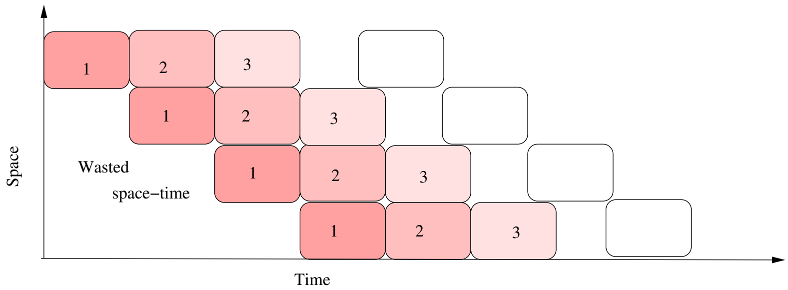


Fig7 D−Blast encoding−numerals represent layers belonging to the same codeword

3.5 ST coding for intermediate rates $(1 \leq r_s \leq M_T)$

It is also possible to trade-off rate of transmission and reliability of transmission. The design metrics (rank and determinant) are not also directly related to the capacity of encoding scheme.

The ST encoding can be viewed as an operator on the channel to yield a new effective channel whose capacity is computed. Taking into consideration the effective channel capacity as one more metrics in addition to the rank and determinant criteria, a more powerful coding schemes could be conceived. There are some new approaches in this direction, eg, the linear dispersion framework proposed by [Hassib ad Hochwald 2001], symbols are spread across time and space through matrix modulation and superposition with the objective of ergodic capacity maximization.

$$\bar{C} = \max_{Tr(\mathcal{X}^H\mathcal{X})=M_T T} \frac{1}{T}\mathcal{E}\{log_2 det(I_{M_R T} + \frac{\rho}{M_T}\mathcal{H}\mathcal{X}\mathcal{X}^H\mathcal{H}^H)\} \quad (16)$$

3.5.1 Signal Model

$N \times 1$ vector $\mathbf{s}$ of N complex data symbols is modulated by $M_T \times N$ code matrix, and transmitted over the $M_R \times M_T$ channel H for each symbol period. Assume there are T distinct code matrices and at time $1 \le k \le T$ signal $\mathbf{X}[k]\mathbf{s}$
(kth code matrix) is transmitted.

The received symbol vector at time instant k will be:

$$\mathbf{y}[k] = \sqrt{\frac{E_s}{M_T}}\mathbf{H}\mathbf{X}[k]\mathbf{s} + \mathbf{n}[k] \quad (17)$$

The code design involves identifying the matrices $\mathbf{X}[k]$ that constitute the code. The matrices are computed to maximize both diversity and ergodic capacity. Similar analysis like presented earlier can apply here also to compute the PEP.

4 ST coding for frequency selective channels

4.1 Signal model

Assuming the the channel between the ith transmit antenna and jth receive antenna is frequency selective, The symbol-sampled baseband impulse response can be denoted by: $h_{i,j}[l], (l = 0, \ldots, L-1)$

As in the flat fading case it is also assumed that there is no channel knowledge at the transmitter and full channel knowledge with ML decoding at the receiver.

$$\mathbf{y}[k] = \sqrt{\frac{E_s}{M_T}} \begin{bmatrix} \mathbf{h}_{1,1} & \dots & \mathbf{h}_{1,M_T} \\ \vdots & \vdots & \vdots \\ \mathbf{h}_{M_R,1} & \dots & \mathbf{h}_{M_R,M_T} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1[k] \\ \vdots \\ \mathbf{s}_{M_T}[k] \end{bmatrix} + \mathbf{n}[k] \qquad (18)$$

4.2 ST codeword design criteria

The codeword construction criterion to obtain full diversity is similar to the flat fading case. The codeword has in effect $\mu = \mathbf{M_T L}$ virtual antennas.

Although there are $M_T L$ virtual antennas, the additional structure imposed may prevent these codes from exploiting full spatio-temporal diversity equal to $M_T M_R L_{eff}$.

References

[1] Paulraj, N. Rohit, G. Dhananjay. Introduction to Space-Time Wireless Communications. Cambridge university press, 2003

[2] B. R. John, L. A. Edward. Digital Communication, third edition Kluwer Academic Publisher, 2004