Multi User Detection I

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January 12, 2005
Outline

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Overview Multiple Access Communication

- **Random Multi Access** (e.g. ALOHA) one user can work simultaneously, otherwise collision occur.

- **FDMA** many user can work simultaneously, but in non overlapping frequency bands.

- **TDMA** many user in one frequency band, but one user per time.

- **CDMA** many user simultaneously in the same frequency band. The user are separated by orthogonal or *non orthogonal signature waveforms*. 
Access Techniques

FDMA

TDMA

CDMA
Motivation: What is MU Detection?

\[ y_k = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k \]

- first term: desired information
- MAI: Multi Access Interference
- n: noise

Cross correlation:

\[ \rho_{ij} = \langle s_i, s_j \rangle = \int_0^T s_i(t) s_j(t) \, dt \]

\( s_i \) ... signature waveform of the \( i^{th} \) user
Where MU Detection can be used?

**Answer:** For nonorthogonal multiple access due to design or due to unwanted crosstalk!

- nonorthogonal CDMA
- TDMA with *Multipath*
- bundle of twisted pair
- magnetic recording in adjacent tracks
**Direct Sequence (DS) SS System**

Power spectrum of $d(t)$

$\eta = \frac{P}{2B}$

$B \approx \frac{1}{T_s}$

Data waveform $d(t) \cdot c(t)$

Power spectrum of $d(t) \cdot c(t)$

$W \approx \frac{1}{T_c}$

FN sequence generator

$T_c = T_s / N$

$1$
Properties of Random Binary Sequences
(Codes in CDMA)

**Autocorrelation** only one single peak

**Cross-correlation** should be low

**Frequency Spectrum** the spread frequency should be flat
Orthogonal / Non Orthogonal

- **Orthogonal code e.g. WALSH**
  - bad spreading properties
  - more peaks in the autocorrelation
  - Orthogonality hold only for perfect synchronized and no multi-path

- **Non Orthogonal code e.g. Gold, Kasami**
  - good spreading properties
  - single peak in the autocorrelation
  - limited (low) cross-correlation
Synchronous/Asynchronous CDMA Model

\[ T = N T_c \]

User 1
User 2
User 3

\( i-1 \)  \( i \)  \( i+1 \)

User 1
User 2
User 3
User K

\( \tau_1 = 0 \)
\( \tau_2 \)
\( \tau_3 \)
\( \tau_K \)
Basic Synchronous CDMA Model (K-user)

\[ y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t), \quad t \in [0, T] \]

- \( s_k(t) \) deterministic signature waveform assigned to the \( k^{th} \) user, with normalized energy
- \( A_k \) amplitude of the \( k^{th} \) user
- \( b_k \in \{-1, +1\} \) is the bit transmitted by the \( k^{th} \) user
- \( n(t) \) is white Gaussian noise with unit power
Discrete Time Synchronous CDMA Model (K-user)

Matched Filter Output

\[ y_1 = \int_0^T y(t)s_1(t) \, dt \]

\[ y_k = \int_0^T y(t)s_k(t) \, dt \]
Cross Correlation

\[ \rho_{ij} = \langle s_i, s_j \rangle = \int_{0}^{T} s_i(t) s_j(t) \, dt \]

Cross correlation matrix for example in the 3-user case:

\[
R = \begin{bmatrix}
1 & \rho_{12} & \rho_{13} \\
\rho_{12} & 1 & \rho_{23} \\
\rho_{13} & \rho_{23} & 1
\end{bmatrix}
\]
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\[ y_k = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k \]

As a Matrix:

\[ y = RAb + n \]

\[ R = \text{normalized cross correlation} \]
\[ y = [y_1, y_2, \ldots, y_k]^T \]
\[ b = [b_1, b_2, \ldots, b_k]^T \]
\[ A = \text{diag} (A_1, A_2, \ldots, A_k) \]
\[ n = \text{n is zero mean Gaussian random vector} \]
Asynchronous Model
Cross correlation

\[ \rho_{kl}(\tau) = \int_{\tau}^{T} s_k(t) s_l(t - \tau) \, dt \]

\[ \rho_{lk}(\tau) = \int_{0}^{\tau} s_k(t) s_l(t + T - \tau) \, dt \]

\[ \rho_{kl} := \rho_{kl}(\tau_l - \tau_k) \]
Basic Asynchronous CDMA Model

Assuming (without losing generality) that a user sends data packets with $2M + 1$ bits.

continuous time CDMA Model for K user:

$$y(t) = \sum_{k=1}^{K} \sum_{i=-M}^{M} A_k b_k[i] s_k(t - iT - \tau_k) + \sigma n(t)$$

$$t \in [-MT, MT + 2T]$$
special case:

\[ A_1 = \ldots = A_K = A \]
\[ s_1 = \ldots = s_K = s \]
\[ \tau_k = \frac{(k - 1)T}{K} \]

\[ y(t) = \sum_{k=1}^{K} \sum_{i=-M}^{M} A b_k[i] s(t - iT - (k - 1)T/K) + \sigma n(t) \]

single user channel with ISI:

\[ y(t) = \sum_{j} A b[j] s(t - jT/K) + \sigma n(t) \]
ISI as an asynchronous CDMA channel with 4 users
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Discrete Time Asynchronous CDMA Model (K-user)

\[ \tau_1 \leq \tau_2 \ldots \leq \tau_k \]

Output of the matched Filter:

\[ y_k = A_k b_k[i] + \sum_{j<k} A_j b_j[i+1] \rho_{k,j} + \sum_{j<k} A_j b_j[i] \rho_{j,k} \]

\[ + \sum_{j>k} A_j b_j[i] \rho_{k,j} + \sum_{j>k} A_j b_j[i-1] \rho_{j,k} + n_k[i] \]
Asynchronous Cross correlation Matrix

Cross correlation matrix for example in the 3-user case:

\[
R[0] = \begin{bmatrix}
1 & \rho_{12} & \rho_{13} \\
\rho_{12} & 1 & \rho_{23} \\
\rho_{13} & \rho_{23} & 1
\end{bmatrix}
R[1] = \begin{bmatrix}
0 & \rho_{21} & \rho_{31} \\
0 & 0 & \rho_{32} \\
0 & 0 & 0
\end{bmatrix}
\]

Time discret asynchronous model in matrix form:

\[
\]
Matched Filter for Single User

\[ y(t) = A \cdot b \cdot s(t) + \sigma \cdot n(t), \quad t \in [0, T] \]

We want to find a \textbf{linear} Filter, which minimize the probability of error.

\[ \hat{b} = sgn(\langle y, h \rangle) = sgn \left( \int_{0}^{T} y(t)h(t)dt \right) \]

due to linearity of decision statistic

\[ Y = \langle y, h \rangle = \underbrace{Ab \langle s, h \rangle}_{\text{signal}} + \underbrace{\sigma \langle n, h \rangle}_{\text{noise}} \]
Some Properties:

If \( h \) is finite-energy deterministic signal and \( n(t) \) is white noise with unit spectral density, then:

1. \( E \left[ \langle n, h \rangle \right] = 0 \)
2. \( E \left[ \langle n, h \rangle^2 \right] = \| h \|^2 \)
3. If \( n(t) \) is a Gaussian process, then \( \langle n, h \rangle \) is a Gaussian random variable.
Maximize SNR of $Y$

$$\max \limits_{h} \frac{A^2 (\langle s, h \rangle)^2}{\sigma^2 \|h\|^2}$$

Cauchy-Schwarz

$$(\langle s, h \rangle)^2 \leq \|h\|^2 \|s\|^2$$

$$\frac{A^2 (\langle s, h \rangle)^2}{\sigma^2 \|h\|^2} \leq \frac{A^2}{\sigma^2} \|s\|^2$$

with equality if and only if $h$ is a multiple of $s$!

The matched filter can be implemented either as a correlator (mult. with $s(t)$ and integration) or by a linear filter with an impulse response $s(T-t)$ sampled at time $T$. 
Proof: Hypothesis Testing Problem:

\[ H_1 : Y \sim f_{Y|1} = N(A \langle s, h \rangle, \sigma^2 \| h \|^2) \]
\[ H_{-1} : Y \sim f_{Y|-1} = N(-A \langle s, h \rangle, \sigma^2 \| h \|^2) \]
Probability of Error

\[ P = \frac{1}{2} \int_{-\infty}^{\infty} f_{Y|\parallel h\parallel}(-v) \, dv + \frac{1}{2} \int_{-\infty}^{0} f_{Y|\parallel h\parallel}(v) \, dv \]

\[ = \frac{1}{2} \int_{\frac{A\langle s, h\rangle}{\sigma \parallel h\parallel}}^{\infty} \frac{1}{\sqrt{2\pi \sigma \parallel h\parallel}} \exp \left( -\frac{v^2}{2\sigma^2 \parallel h\parallel^2} \right) \, dv + \frac{1}{2} \int_{-\frac{A\langle s, h\rangle}{\sigma \parallel h\parallel}}^{-\infty} \frac{1}{\sqrt{2\pi \sigma \parallel h\parallel}} \exp \left( -\frac{v^2}{2\sigma^2 \parallel h\parallel^2} \right) \, dv \]

\[ = \int_{\frac{A\langle s, h\rangle}{\sigma \parallel h\parallel}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{v^2}{2} \right) \, dv = Q \left( \frac{A\langle s, h\rangle}{\sigma \parallel h\parallel} \right) \]
Probability of Error (2)

\[ P = Q \left( \frac{A \langle s, h \rangle}{\sigma \| h \|} \right) \]

The Q function is monotonic decreasing. We assume that argument of Q-function is non-negative and so we can maximize the square of the argument to minimize the error probability. And that we already did before.

\[ P^C = Q \left( \frac{A}{\sigma} \right) \]
References

- Multiuser Detection, Serio Verdu

- Spread Spectrum Technique and its Application to DS/CDMA, Bernard H. Fleury and Alexander Kocian
  http://kom.aau.dk/project/sipcom/sites/sipcom9/courses/CDMA/SS_all_2.pdf

THE END