Multi User detection I

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Abstract

This document should give the reader an introduction to "Multi User detection" MUD. It starts with a short look at multi access communication, especially CDMA. We will specifically deal with orthogonal and nonorthogonal design. Then we go on with an outline, what Multi User detection is and where it can be used. The document also provides an introduction to CDMA and its channel models. Finally we will derive a matched filter for a single user in a CDMA channel.

Contents

1 Overview Multiple Access Communication

When the number of potential users is much greater than the number of simultaneously active users at any given time, it makes sense to use multi access communication. There are several ways to provide multiple access in communication.

1.1 Random Multi Access e.g. ALOHA

Random Multi Access is one of the approaches to dynamic channel sharing. The main idea is, that, if a user has to transmit a data packet it will do it. If two users are trying to send a packet a collision will happen and both users have (after a random waiting time) to retransmit their packet. Random Access works well if it is very likely that only one user has to transmit something at the same time.

1.2 FDMA Frequency Division Multiple Access

The idea of FDMA is that users overlap in space and time but not in the frequency. FDMA allows multiple users to share a physical communication channel by modulating the users baseband signal on different frequency carrier waves and add them together. The frequency bands of the different users are non overlapping, see figure 1. To be sure that the user does not interfere, a guard band is given between the users. Due to the users do not interfere with each other the design is called orthogonal.
If a user only transmits something once in a while, it would be a waste of capacity to give each user an own frequency band. To provide dynamic channel sharing in FDMA a subchannel has to be used as a reservations channel.

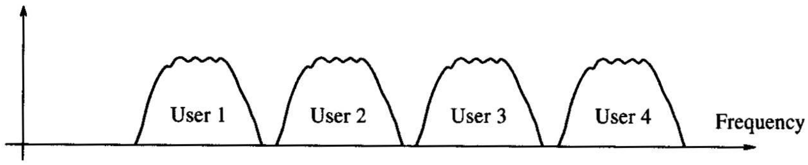


Figure 1: FDMA Frequency bands

1.3 TDMA Time Division Multiple Access

TDMA is very similar to FDMA. In TDMA the users overlap in space and frequency but not in the time. One Channel is divided into several non overlapping *time slots* (orthogonal channels) see figure 2. To provide dynamic channel sharing also a reservations channel has to be used.

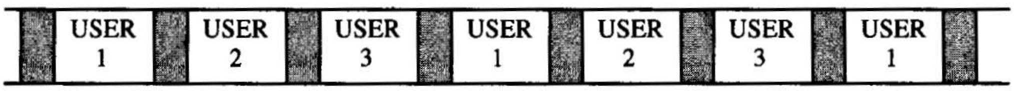


Figure 2: TDMA time slots for 3 users

1.4 CDMA Code Division Multiple Access

In CDMA the users overlap in space, time and frequency, all users transmit in the same bandwidth simultaneously without harmful interference. In order to avoid harmful interference, the data of a single user is multiplied with a a unique signature waveform. The signature waveform has a much wider frequency spectrum than the user data of a single user. For this reason this concept is called "spread spectrum systems". The users are separated by orthogonal or non orthogonal signature waveforms (Codes). It will be shown later, that nonorthogonal signature waveforms have advantages over the orthogonal's. The users share the Signal-to-Noise-Ratio.
Figure 3 shows an overview of how FDMA, TDMA and CDMA divide one channel into subchannels.

2 Motivation

2.1 What is MU Detection?

As we will see in section 2.2 MU Detection can be applied in many cases, but now we want to study MUD in CDMA. For nonorthogonal CDMA a

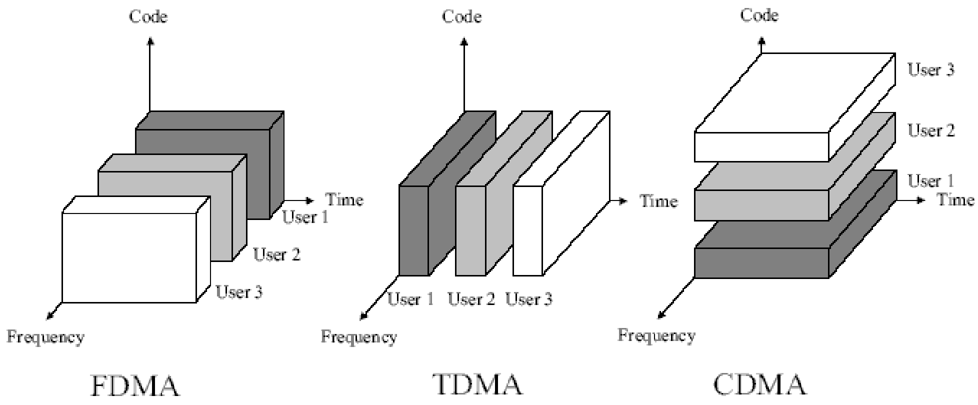


Figure 3: ACCESS TECHNIQUES

matched Filter is not the best detector anymore. In nonorthogonal CDMA the inner product or cross correlation

$$\rho_{1,2} = \langle s_1, s_2 \rangle = \int_0^T s_1(t) s_2(t) dt \neq 0 \tag{1}$$

is not zero anymore. For this reason the users will interfere, which is called Multi Access Interference(MAI) or cross talk. Below there is the equation of the received signal after the matched filter for the k^{th} user.

$$y_k = \underbrace{A_K b_k}_{1stterm} + \underbrace{\sum_{j \neq k} A_j b_j \rho_{jk}}_{MAI} + \underbrace{n_k}_{n} \tag{2}$$

The equation has three parts. The first part is the desired information. The second term is the Multi Access Interference and the third term is the noise. In MUD we try to build a filter which chancel the second term. We want to build a detector which cancels the Multi Access Interference.

2.2 Where can MU Detection be used?

Answer: MU Detection can be applied always for multiple access when nonorthogonality occurs. The nonorthogonality may come from the design as in CDMA or by unintentional cross talk. Here is a list of some applications for which MUD makes sense:

- CDMA
- TDMA due to *Multipath*

- bundle of twisted pair
- magnetic recording in adjacent tracks

As already mentioned CDMA can have nonorthogonality already from the design, but Multi Access Interference can also occur in orthogonal CDMA. The orthogonality property holds only if the users are perfectly synchronized and there is no multipath.
TDMA can be seen as a special case of orthogonal CDMA in which the signature waveforms do not overlap in the time domain. Figure 4 shows the signature waveforms of TDMA.

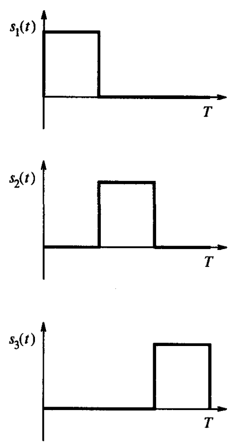


Figure 4: Signature Waveforms in TDMA

Another classical example for MUD is a bundle of twisted pair with radial and capture energy. Digital transmission carried by each pair is subject to contamination due to the superposition of the attenuated transmission carried by neighboring pairs. A similar phenomenon occurs in adjacent tracks in magnetic recording.

3 Overview of DS/CDMA systems

3.1 Concept

The principle of CDMA is, that we multiply the data with signature waveforms. The signature waveforms have normally a much wider bandwidth than the data. The data bandwidth is spread into a bigger one. The signature waveforms can have any shape with a low cross correlation, but now

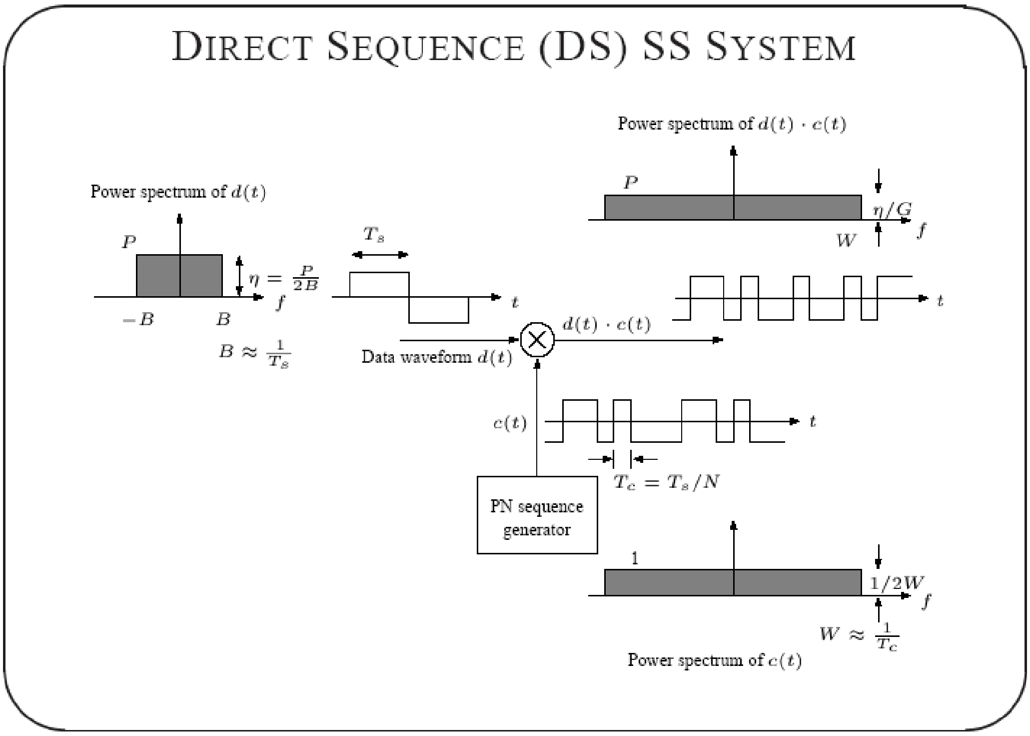


Figure 5: DS CDMA System

we want to focus on Direct Sequence (DS); This is the most popular Spread Spectrum Technique . In DS the signature waveform is a Pseudo Random Noise Code, which can be easily generate by a linear feedback shift register.

3.2 Orthogonal / Nonorthogonal

Orthogonal CDMA has the big advantage, that the cross correlation is zero and for this reason we expect that there is no MAI. But orthoganality has also several drawbacks.

- In an orthogonal CDMA the users need to be perfectly synchronous.
- Even if we have perfectly synchronized, multipath can destroy the orthogonal property.
- The number of simultaneous users in orthogonal CDMA is $2TB$ of the signature waveform.
  T ... duration of the signature waveform
  B ... approximately the bandwidth of the signature waveform (e.g. 99% of the energy)
- In section 3.3 we will show some more drawbacks of DS-CDMA.

Nonorthogonal CDMA does not suffer from the limitations above

3.3 RANDOM BINARY SEQUENCES

Codes used in CDMA can be characterized by:

- chip waveform One period of the data sequence is called BIT and one period of the PN sequence is called CHIP.
- numbers of chips per bit N
- binary sequence of of the length N:$(c_1, c_2, \ldots, c_N]$

The number of chips per bit has not to be equal to the length of the code, but for the rest of the document we assume that.
Codes used in CDMA should have the following properties:

Autocorrelation should be only one single peak, see figure 6.

Let $a = (c_0, c_1, \ldots, c_{N-1})$,$b = (c_0, c_1, \ldots, c_{N-1})$, $c_n \in \{-1, +1\}$ denote two binary sequences of length N.
Let $a^{(l)}$ denote the l-times cyclic right-shifted version of a.

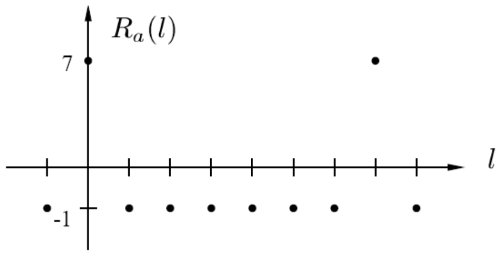


Figure 6: Autocorrelation

$$R_a(l) = \sum_{n=0)}^{N-1} a_n a_n^{(l)} \tag{3}$$

Properties of $R_a(l)$:

$$R_a(0) = N$$
$$|R_a(l)| \leq R_a(l)$$

Cross-correlation should be low.

$$R_{a,b}(l) = \sum_{n=0)}^{N-1} a_n b_n^{(l)} \tag{4}$$

Frequency Spectrum the spread frequency band should be flat.

Balance The number of 1's should be the same as the -1's. The DC Level should be 0 to suppress the carrier.

3.3.1 Orthogonal code e.g. WALSH

- bad spreading properties
- more peaks in the autocorrelation
- Orthogonality holds only when perfectly synchronized and there is no multi-path

3.3.2 Non Orthogonal code e.g. Gold, Kasami

- good spreading properties
- single peak in the autocorrelation
- limited (low) cross-correlation

4 CDMA Channel Model

4.1 Basic Synchronous CDMA Model (K-user)

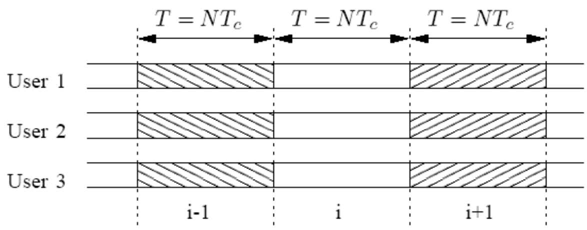


Figure 7: synchronous CDMA

Synchronous CDMA means that all users have got the same signature sequence duration T and that all user start to transmit a new bit at the same time, see figure 7.
The equation for a continuous time K-user channel model is:

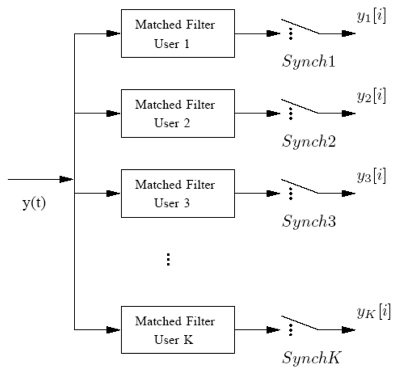
$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t),\ t \in [0, T] \tag{5}$$

where

- T is the inverse of the data rate
- $s_k(t)$ deterministic signature waveform assigned to th k^{th} user, with normalized unit energy

$$\|s_k\|^2 = \int_0^T s_k(t) d\,t = 1 \tag{6}$$

- A_k amplitude of the k^{th} user
- $b_k \in \{-1, +1\}$ is the bit transmitted by the k^{th} user
- $n(t)$ is white Gaussian noise with unit power.



$$
\begin{aligned}
y_1 &= \int_0^T y(t)s_1(t)\,dt \\
&\vdots \\
y_k &= \int_0^T y(t)s_k(t)\,dt
\end{aligned}
$$

Figure 8: Bank of Matched Filters

4.2 Discrete Time Synchronous CDMA Model

One way of converting the received waveform into a discrete-time process is to pass it through a bank of matched filters (figure 8), each matching the signature waveform of a different user. We can do this without losing relevant information. In synchronous case, the output of the k^{th} user is

$$y_k = A_K b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k \tag{7}$$

where

$$n_k = \sigma \int_0^T n(t)s_k(t)dt \tag{8}$$

We can write the outputs of all users in matrix form:

$$y = RAb + n \tag{9}$$

where

$$
\begin{aligned}
R &= \text{normalized cross correlation} \\
y &= [y_1, y_2, \ldots, y_k]^T \\
b &= [b_1, b_2, \ldots, b_k]^T \\
A &= diag\,(A_1, A_2, \ldots, A_k)
\end{aligned}
$$

Equation 10 is the cross correlation between the users i and j.

$$\rho_{ij} = \langle s_i, s_j \rangle = \int_0^T s_i(t)s_j(t)\,dT \tag{10}$$

Equation 11 is an example of a Cross correlation matrix in the 3-user case:.

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \tag{11}$$

4.3 Basic Asynchronous CDMA Model

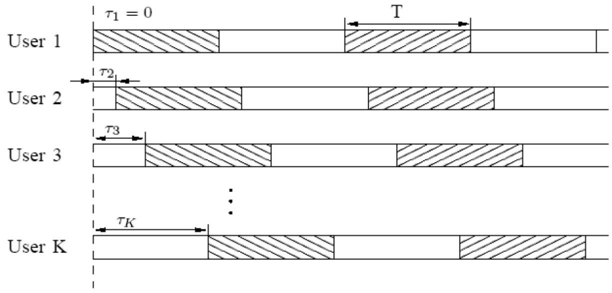


Figure 9: Asynchronous CDMA

Asynchronous CDMA means that all users have got the same signature sequence duration T, but the user starts to transmit a new bit at a random time, see figure 9. The time-difference between the users is called offset $\tau_k \in [o, T)$. Without losing generality we assume that $\tau_0 = 0$ and that

$$\tau_1 \leq \tau_2 \ldots \leq \tau_k.$$

This can be easily achieved by renumbering the users. Furthermore we assume (without losing generality) that a user sends data packets with $2M + 1$ bits. So the continuous time model is

$$\begin{aligned} y(t) &= \sum_{k=1}^{K} \sum_{i=-M}^{M} A_k b_k[i] s_k(t - iT - \tau_k) + \sigma n(t) \\ & \quad t \in [-MT, MT + 2T]\,. \end{aligned} \tag{12}$$

4.3.1 Special Case Asynchronous CDMA Model

In this special case:

$$A_1 = \ldots = A_K = A \tag{13}$$

$$s_1 = \ldots = s_K = s \tag{14}$$

$$\tau_k = \frac{(k-1)T}{K} \tag{15}$$

$$
\begin{aligned}
y(t) &= \sum_{k=1}^{K} \sum_{i=-M}^{M} Ab_k[i]s(t - iT - (k-1)T/K) + \sigma n(t) \\
&= \sum_{j} Ab[j]s(t - jT/K) + \sigma n(t)
\end{aligned} \tag{16}
$$

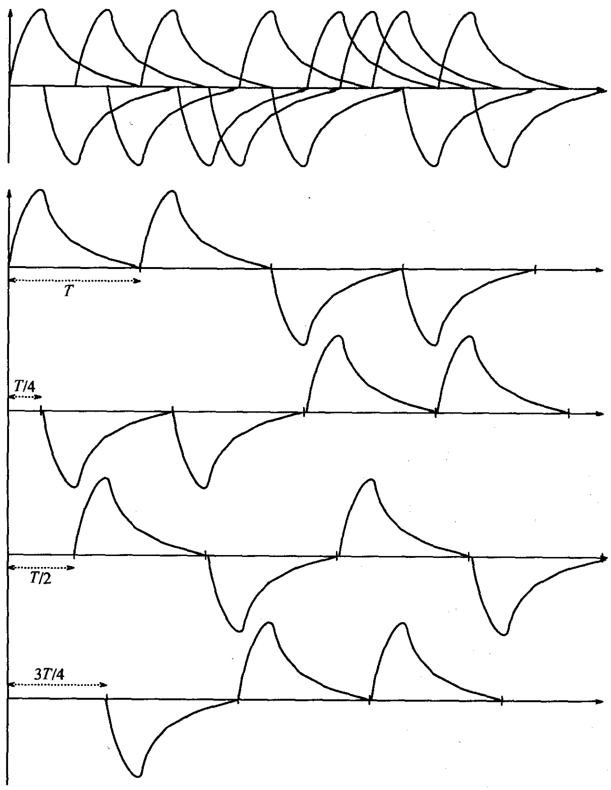


Figure 10: ISI as a asynchronous CDMA channel.

Equation 16 is , in fact the standard single-user white Gaussian noise channel with signals subject to intersymbol interference(ISI). Every bit overlaps with a fixed number finite number of neighboring bits. Figure 10 shows an example where every symbol overlaps with 3 preceding symbols and 3 succeeding symbols. This is equivalent to an asynchronous CDMA channel with 4 "users" (each free of ISI) where the signature waveform are common to all users ant the offsets are $\tau_1 = 0$, $\tau_2 = T/4$, $\tau_1 = T/2$ and $\tau_2 = 3T/4$. Each "user" carry every fourth bit or the original data stream.

4.4 Discrete Time Asynchronous CDMA Model

As in the synchronous case we pass the received waveform into a bank of matched filters to get the Discrete Time Model. The equation is

$$\begin{aligned} y_k = A_k b_k[i] + \sum_{j<k} A_j b_j[i+1]\rho_{kj} + \sum_{j<k} A_j b_j[i]\rho_{jk} \\ + \sum_{j>k} A_j b_j[i]\rho_{kj} + \sum_{j>k} A_j b_j[i-1]\rho_{jk} + n_k[i].. \end{aligned} \quad (17)$$

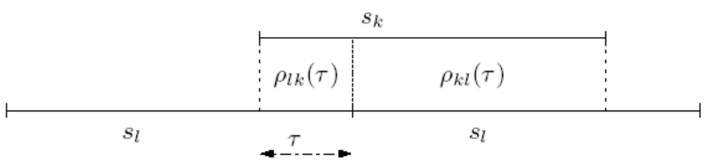


Figure 11: Asynchronous Cross correlation

Due to the offset in asynchronous CDMA the synchronous cross correlation is not sufficient anymore. In asynchronous CDMA we must calculate two cross correlations between every pair of signature waveforms that depend on the offset between the signals. If $k < l$, then we denote

$$\rho_{kl}(\tau) = \int_{\tau}^{T} s_k(t)s_l(t-\tau)\, dT \quad (18)$$

$$\rho_{lk}(\tau) = \int_{0}^{\tau} s_k(t)s_l(t+T-\tau)\, dT. \quad (19)$$

where $\tau \in [0, T)$. the subindex of $\rho_{kl}(\tau)$ denotes the "left" signal in the correlation, see figure 11. To save writing work and make it more readable, we will use the following notation:

$$\rho_{kl} := \rho_{kl}(\tau_l - \tau_k)$$

We can also write the different correlations in matrix form. Here is an example for a three users case:

$$R[0] = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \quad R[1] = \begin{bmatrix} 0 & \rho_{21} & \rho_{31} \\ 0 & 0 & \rho_{32} \\ 0 & 0 & 0 \end{bmatrix} \quad (20)$$

The time discrete asynchronous model in matrix form reads

$$y[i] = R^T[1]\, Ab[i+1] + R[0]\, Ab[i] + R[1]\, Ab[i-1] + n[i] \quad (21)$$

5 Matched Filter for single user

Now we study the single-user version of a basic CDMA channel. We have only one user, synchronization happens automatically and the channel becomes

$$y(t) = A \cdot b \cdot s(t) + \sigma \cdot n(t),\ t \in [0, T]\,. \tag{22}$$

where the deterministic signal s has unit energy, the noise is white as Gaussian, and $b \in [1, -1]$.
We want to find a **linear Filter**, which minimizes the error probability. Consider a demodulator for the CDMA channel that outputs the *sign* of the correlation of the observed waveform with the deterministic signal h of duration T.

$$\widehat{b} = sgn(\langle y, h \rangle) = sgn\left(\int_0^T y(t)h(t)dt\right) \tag{23}$$

To optimize the choice of h regarding to the Signal-to-Noise-Ratio, we split the equation into a signal part and a noise part. This is possible, because due to linearity of decision statistics vector Y.

$$Y = \langle y, h \rangle = \underbrace{Ab\,\langle s, h \rangle}_{signal} + \underbrace{\sigma\,\langle n, h \rangle}_{noise} \tag{24}$$

The distribution of the noise term is given by the following results, which states the main properties of white noise we use throughout this text.

Proposition 1: *If h and g are finite- energy deterministic signals and n(t) is white noise with unit spectral density, then:*

1. $E\left[\langle n, h \rangle\right] = 0$
2. $E\left[\langle n, h \rangle^2\right] = \|h\|^2$
3. If n(t) is a Gaussian process, then $\langle n, h \rangle$ is a Gaussian random variable
4. $E\left[\langle n, h \rangle \langle n, g \rangle\right] = \langle g, h \rangle$
5. $E\left[\langle n, h \rangle\, n\right] = h$

According to Proposition 1.2 $\sigma\,\|h\|$ is the standard derivation of the contribution of the noise to the decision statistic in equation 24. Thus we would like to choose h to make $\langle s, h \rangle$ big and $\sigma\,\|h\|$ small.

$$\max_h \frac{A^2\left(\langle s, h \rangle\right)^2}{\sigma^2\,\|h\|^2} \tag{25}$$

Now we use the Cauchy-Schwarz in-equation.

$$(\langle s, h \rangle)^2 \leq \|h\|^2 \|s\|^2 \tag{26}$$

In equation 27 we can chancel $\|h\|^2$. So that maximum SNR is at $\frac{A^2}{\sigma^2} \|s\|^2$ and the Cauchy-Schwarz in-equation tells us also when and only when this value can be reached. Cauchy-Schwarz in-equation has **equality if and only if h is a multiple of s!**

$$\frac{A^2 (\langle s, h \rangle)^2}{\sigma^2 \|h\|^2} \leq \frac{A^2}{\sigma^2} \|s\|^2 \tag{27}$$

If h is a non negative multiple of s

$$h = \alpha s \qquad \alpha > 0$$

the SNR has its maximum value. Equation 23 has become

$$\widehat{b} = sgn(\langle y, \alpha s \rangle) = sgn\left(\int_0^T y(t)s(t)dt\right). \tag{28}$$

This detector (equation 28) is called **matched filter**. The matched filter can be implemented either as a correlator (multiplication of the received waveforms with s followed by integration) or as a linear filter with an impulse response s(T-t) sampled at time T.

5.1 Proof

We have seen that the matched filter is optimal in the sense that it maximizes the SNR. Notice that in this derivation we did not consider the fact that the noise is Gaussian. Proposition 1 and equation 22 lead to the conclusion that the distribution of the decision statistic Y conditioned on b=-1 is Gaussian with mean $-A\langle s, h \rangle$ and the variance $\sigma^2 \|h\|^2$, which is abbreviated as $N(-A\langle s, h \rangle, \sigma^2 \|h\|^2)$. Analogously, b=1 yields $N(A\langle s, h \rangle, \sigma^2 \|h\|^2)$

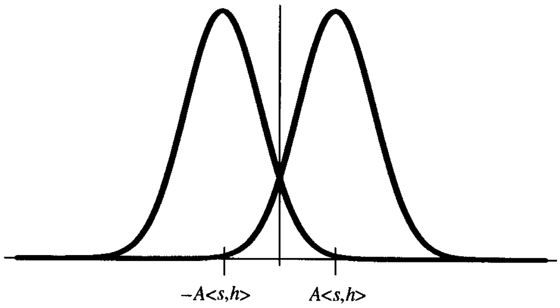


Figure 12: Conditional distributions of Y (B=-1 or 1)

Therefore, we have the binary hypothesis testing problem:

$$H_1 : Y \sim f_{Y|1} = N(A\langle s,h\rangle, \sigma^2 \|h\|^2) \quad (29)$$

$$H_{-1} : Y \sim f_{Y|-1} = N(-A\langle s,h\rangle, \sigma^2 \|h\|^2) \quad (30)$$

The distributions can be seen in figure 12. To make a decision b=1 or b=-1 we put the threshold to zero.

So we can calculate the Probability of Error:

$$\begin{aligned} P &= \frac{1}{2}\int_0^\infty f_{Y|-1}(v)\, dv + \frac{1}{2}\int_{-\infty}^0 f_{Y|1}(v)\, dv & (31) \\ &= \frac{1}{2}\int_{A\langle s,h\rangle}^\infty \frac{1}{\sqrt{2\pi}\sigma\|h\|}\exp\left(-\frac{v^2}{2\sigma^2\|h\|^2}\right) dv + \\ &\quad \frac{1}{2}\int_{-\infty}^{-A\langle s,h\rangle} \frac{1}{\sqrt{2\pi}\sigma\|h\|}\exp\left(-\frac{v^2}{2\sigma^2\|h\|^2}\right) dv & (32) \\ &= \int_{\frac{A\langle s,h\rangle}{\sigma\|h\|}}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right) dv & (33) \\ &= Q\left(\frac{A\langle s,h\rangle}{\sigma\|h\|}\right) & (34) \end{aligned}$$

The Q function is monotonically decreasing, see figure 13. We assume that the argument of the Q-function is non negative and so we can maximize the square of the argument to minimize the error probability. And this is what we already did before in equation 25. Matched filtering and zero-threshold is the best detector for a single user in CDMA with white Gaussian noise. The probability of error is:

$$P^C = Q\left(\frac{A}{\sigma}\right) \quad (35)$$

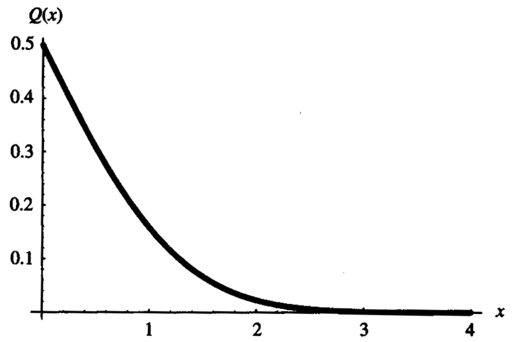


Figure 13: Q-function

6 References

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