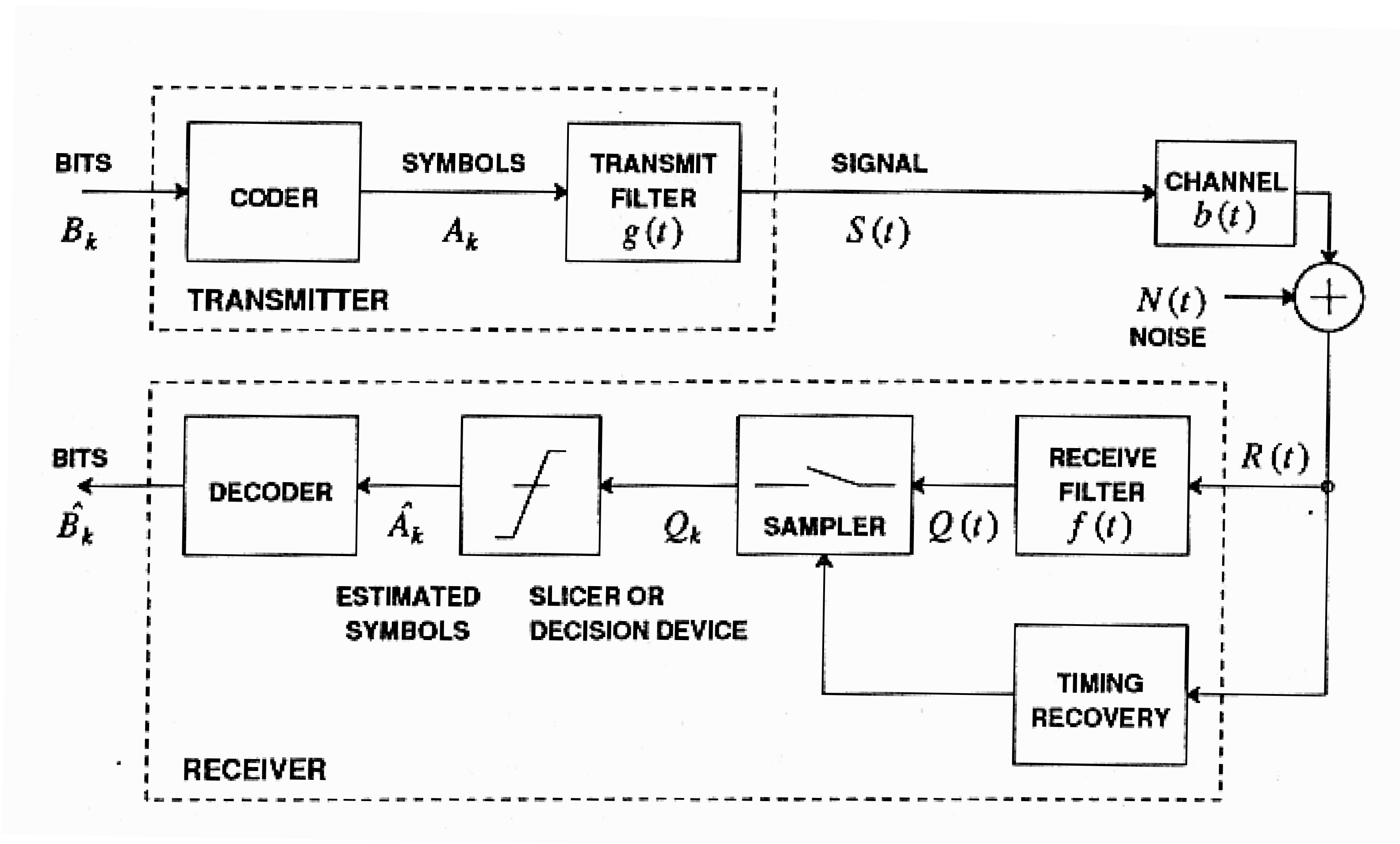
Maximum Likelihood Sequence Detection

- Channel
- ML Detection
- Error Probability

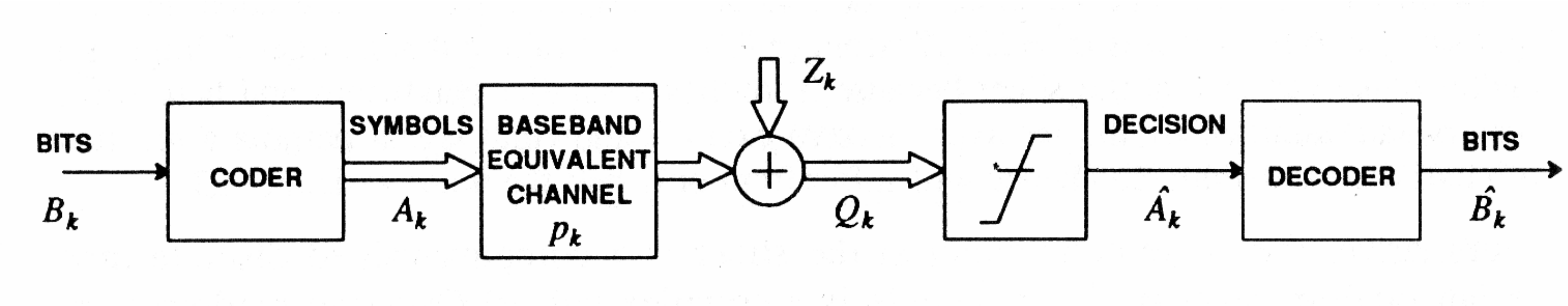
Channel

- Delay Spread
  - *time dispersion*
  - *intersymbol interference (ISI)*.
  - *frequency selective fading*
- Channel Model
  - passband PAM
  - baseband PAM

Channel Model



Discrete-Time Equivalent Channel Model for PAM

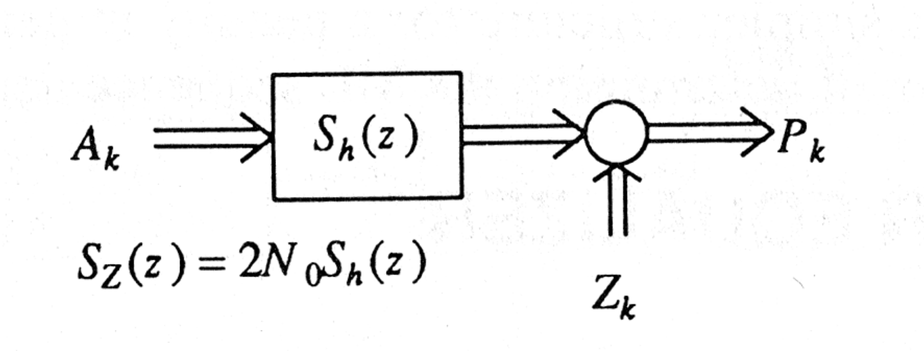


$$P(e^{j\omega T}) = \sum_{m=-\infty}^{\infty} G\left[j\left(\omega - \frac{2\pi}{T}m\right)\right] B_E\left[j\left(\omega - \frac{2\pi}{T}m\right)\right] F\left[j\left(\omega - \frac{2\pi}{T}m\right)\right]$$

$$S_Z(e^{j\omega T}) = \frac{2N_0}{T}\sum_{m=-\infty}^{\infty}\left|F\left(j\left(\omega + m\tfrac{2\pi}{T}\right)\right)\right|^2$$

Matched Filter as Receiver Front End (1)

- matched filter as receive filter
- discrete-time equivalent channel model



Matched Filter as Receiver Front End (2)

- autocorrelation of baseband receive pulse shape h(t)

$$\rho_h(k) = \int_{-\infty}^{\infty} h(t) h^*(t - kT)\, dt$$

- folded spectrum

$$S_h(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} \rho_h(k) e^{j\omega k T} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \left| H\left(j\left(\omega + m \tfrac{2\pi}{T} \right)\right)\right|^2$$

Whitening of the Matched Filter (1)

- matched filter ➔ colored noise

$$S_Z(e^{j\omega T}) = 2N_0 S_h(e^{j\omega T})$$

- spectral factorization

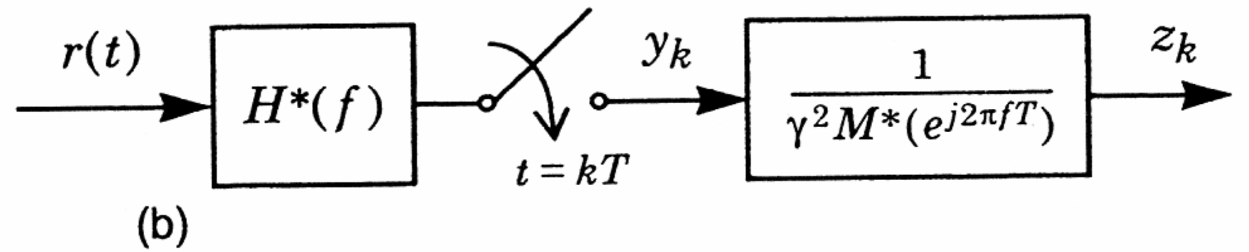
$$S_h(z) = \gamma^2 M(z) M^*(1/z^*)$$

- minimal phase and allpass system

$$S_h(z) = H_{\min}(z) H_{ap}(z)$$

Whitening of the Matched Filter (2)

- equalization with inverse minimal phase filter



- noise process has white power spectrum

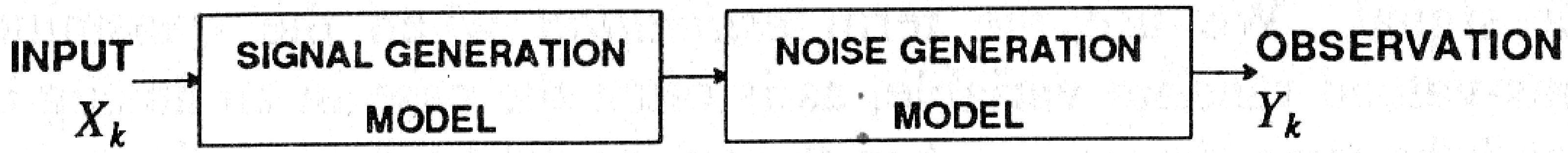
$$S_N(z) = \frac{2N_0}{K}$$

Detection

- ML Detection of a Single Symbol
- ML Detection of a Signal Vector
- ML Detection with Intersymbol Interference
- Sequence Detection
  - Markov Chains
  - Markov Chain Signal Generator
  - The Viterbi Algorithm

Detection

- Estimation ➔ transmitted signal is contiuous-valued
- Detection ➔ transmitted signal is discrete-valued
- Model for detection:



ML Detection of a Single Symbol

- Special case of MAP detector if $p_A(\hat{a}) = const$

$$p_{A|Y}(\hat{a} \mid y) = \frac{p_{Y|A}(y \mid \hat{a})\, p_A(\hat{a})}{p_Y(y)}$$

- ML chooses $\hat{a}\, \varepsilon\, \Omega_A$
- To maximize likelihood $p_{Y|A}(y \mid \hat{a})$
- Measure of the quality $\Pr[error] = \Pr[\hat{a} \neq a]$

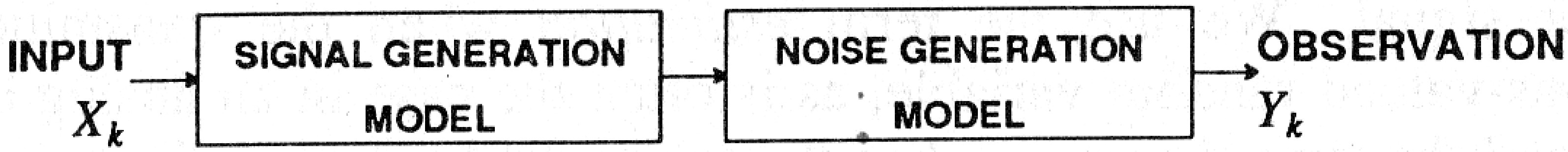
ML Detection of a Signal Vector

- Vector of Symbols
- Maximize $f_{\mathbf{Y}|\mathbf{S}}(\mathbf{y} \mid \hat{\mathbf{s}}) = f_{\mathbf{N}|\mathbf{S}}(\mathbf{y} - \hat{\mathbf{s}} \mid \hat{\mathbf{s}}) = f_{\mathbf{N}}(\mathbf{y} - \hat{\mathbf{s}})$
- Equivalent to maximize

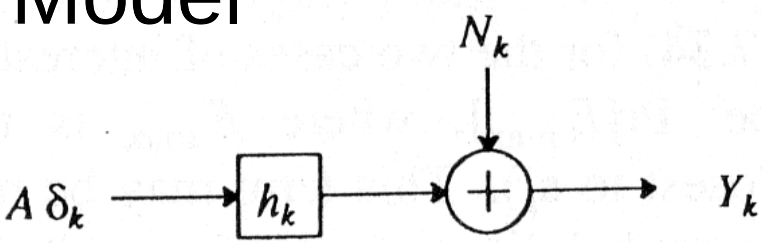
$$f_{\mathbf{N}}(\mathbf{y} - \hat{\mathbf{s}}) = \frac{1}{(2\pi)^{M/2} \sigma^M} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{y} - \hat{\mathbf{s}}\|^2 \right)$$

- Equivalent to minimizing $\|\mathbf{y} - \hat{\mathbf{s}}\|$

ML Detection With Intersymbol Interference (1)



- Generator is LTI filter $h_k, 0 \le k \le M$
- Input single data symbol A
- Model



$$\mathbf{Y} = \mathbf{h}A + \mathbf{N}$$

ML Detection With Intersymbol Interference (2)

- ML minimizes distance $\hat{a}\mathbf{h}$ and observation $\mathbf{y}$

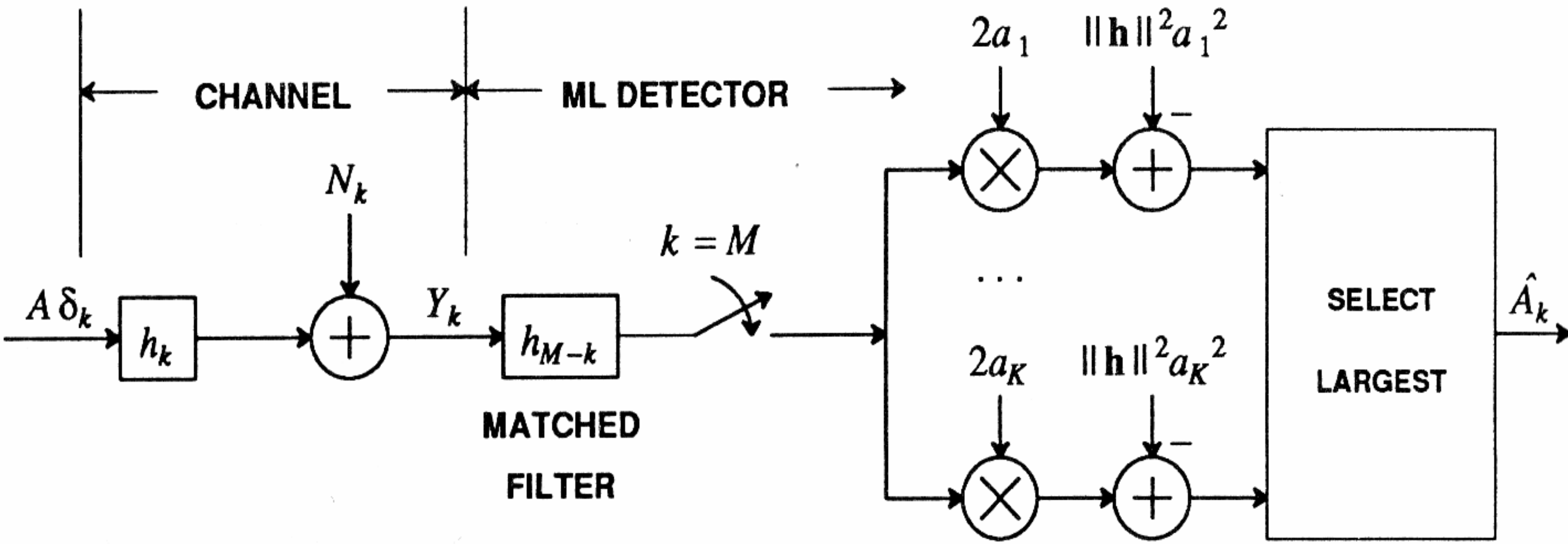
$$\|\mathbf{y}-\mathbf{h}\hat{a}\|^2 = \|\mathbf{y}\|^2 - 2\langle \mathbf{y},\mathbf{h}\rangle \hat{a} + \|\mathbf{h}\|^2 \hat{a}^2$$

- Equivalent to maximizing

$$2\langle \mathbf{y},\mathbf{h}\rangle \hat{a} - \|\mathbf{h}\|^2 \hat{a}^2$$

$$\langle \mathbf{y},\mathbf{h}\rangle = \sum_m y_m h_m = \left[y_k * h_{-k}\right]_{k=0}$$

ML Detection With Intersymbol Interference (3)



ML Detection With Intersymbol Interference (4)

- Exponential complexity
- Message of K M-ary symbols

➔ M^K matched filters

➔ M^K comparisons

Sequence Detection

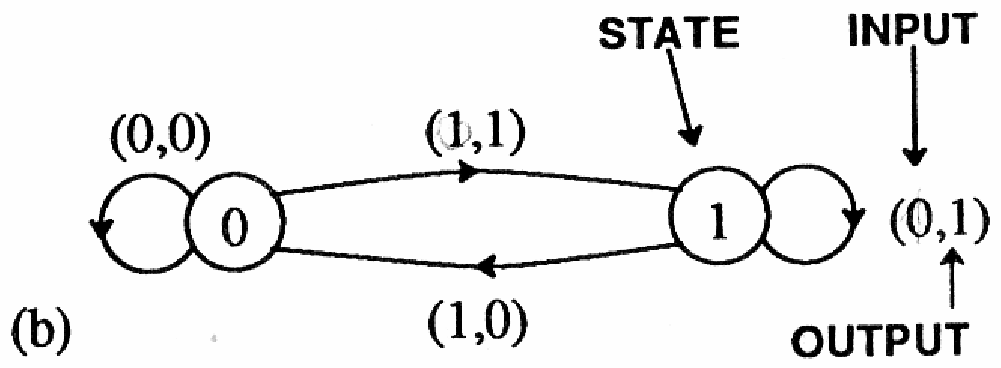
- Markov Chains
- Markov Chain Signal Generator
- The Viterbi Algorithm

Markov Chains (1)

- Independent of past samples

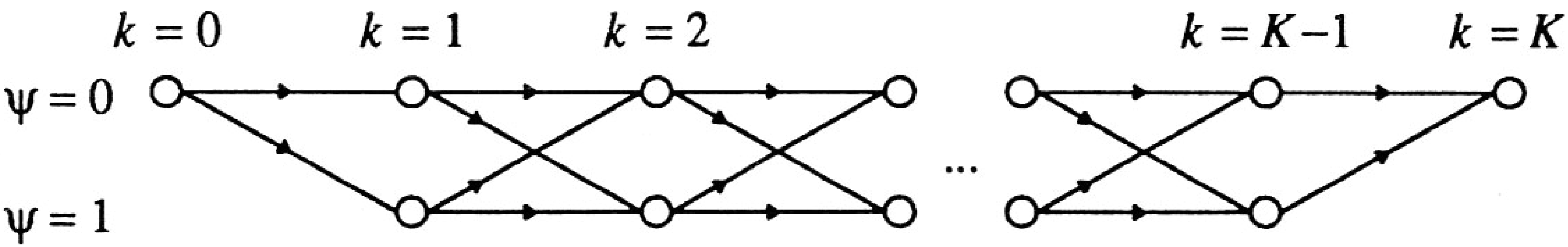
$$p\left(\Psi_{k+1} \mid \Psi_k, \Psi_{k-1}, \ldots\right) = p\left(\Psi_{k+1} \mid \Psi_k\right)$$

- *Homogenous* if independent of k
- State transition diagram



Markov Chains (2)

- Trellis diagram



- Node
- Branch
- Path

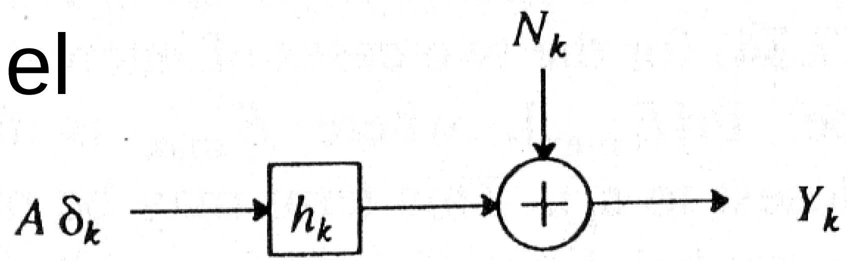
Markov Chain Signal Generator (1)

- Sequence of homogenous Markov chain states Ψ_k
- State transitions $S_k = g(\psi_k, \psi_{k+1})$
- Observation function $g(\psi_k, \psi_{k+1}) = \sum_{i=0}^{M} h_i A_{k-1}$
- State of shift-register

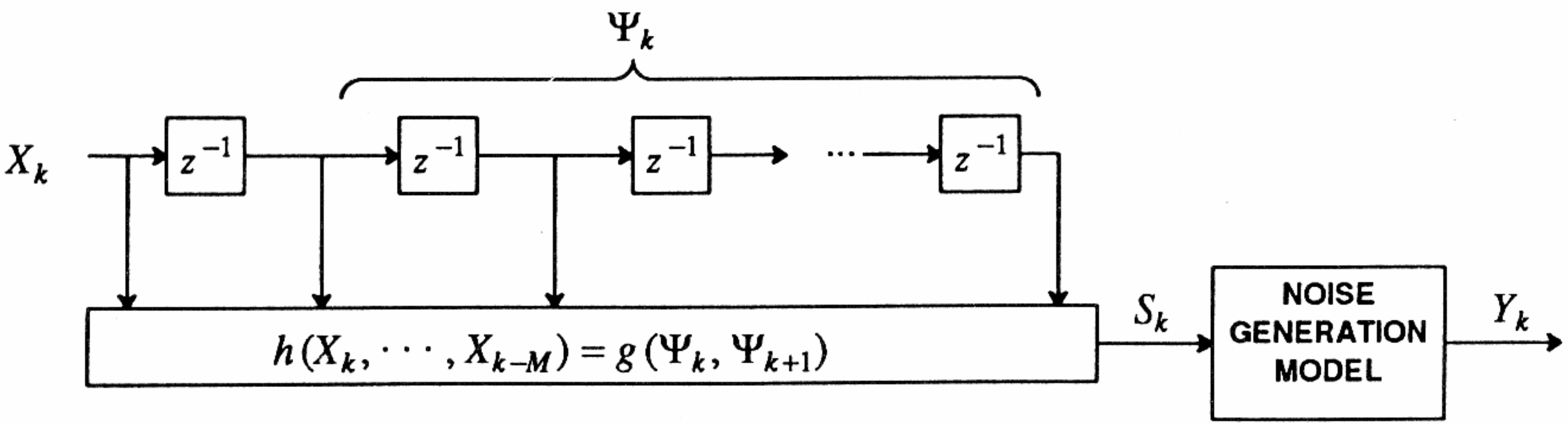
$$\Psi_k = [X_{k-1}, X_{k-2}, \ldots, X_{k-M}]$$

Markov Chain Signal Generator (2)

- ISI Model

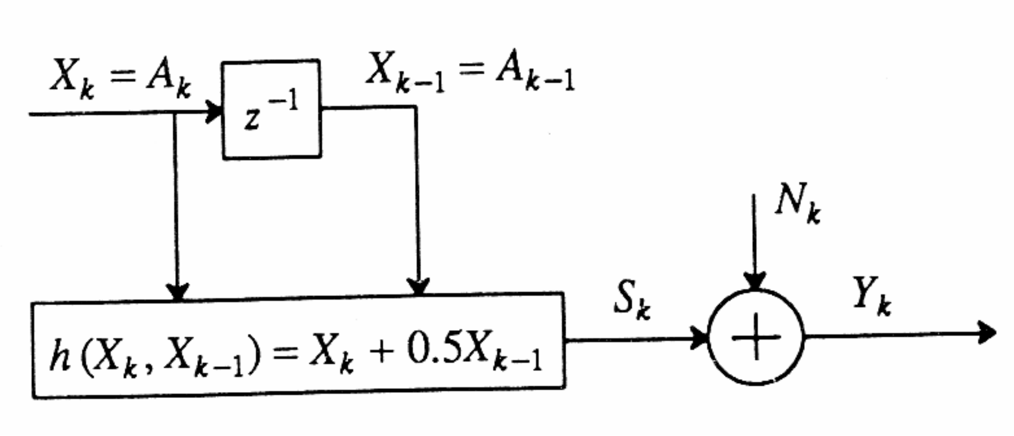


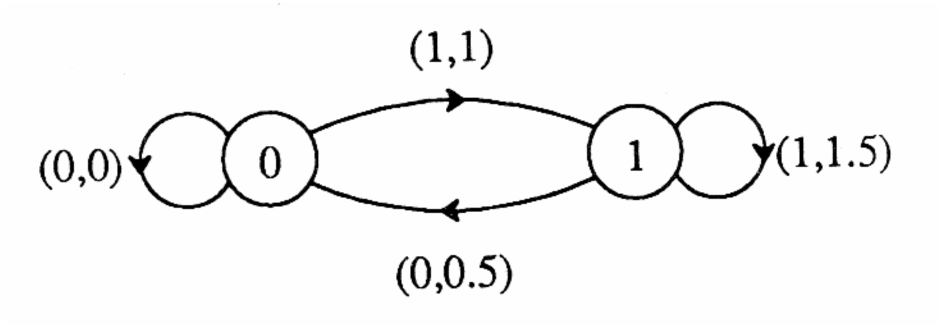
- Shift-register process

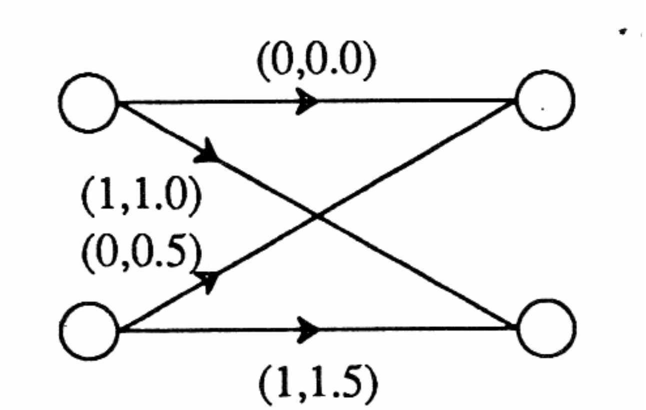


Markov Chain
Signal Generator Example

$$h_k = \delta_k + 0.5\delta_{k-1}$$







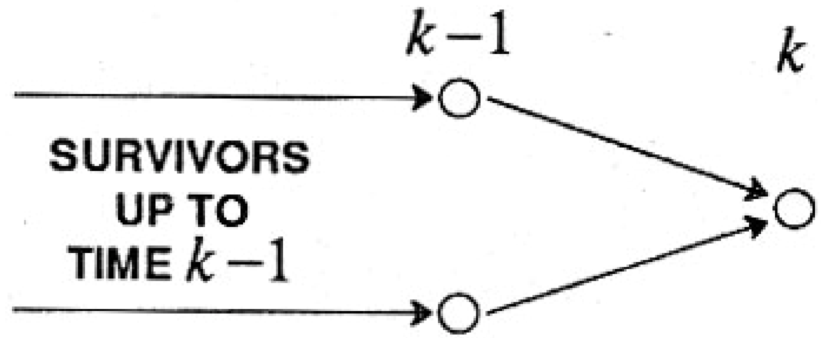
The Viterbi Algorithm (1)

- A. Viterbi of UCLA in 1967
- Homogenous Markov chain
- Linear complexity growing with message length K
- Application for maximization problems

The Viterbi Algorithm (2)

- Sequence of inputs = path through the trellis
- Assign $branch\,metric = \left| y_k - s_k \right|^2$
- Path metric = Σ branch metrics
- Choose lowest path metric = minimize $\left\| \mathbf{y} - \hat{\mathbf{s}} \right\|$

The Viterbi Algorithm (3)



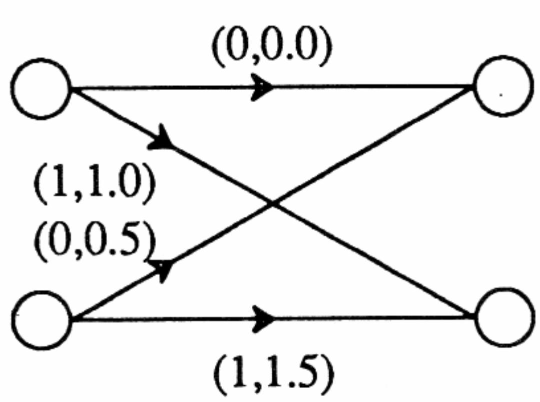
- Survivor path of k-1 = smallest path metric to node k-1
- Only hold survivor path
- For node k choose smallest branch metric + survivor path of k+1

The Viterbi Algorithm Example (1)

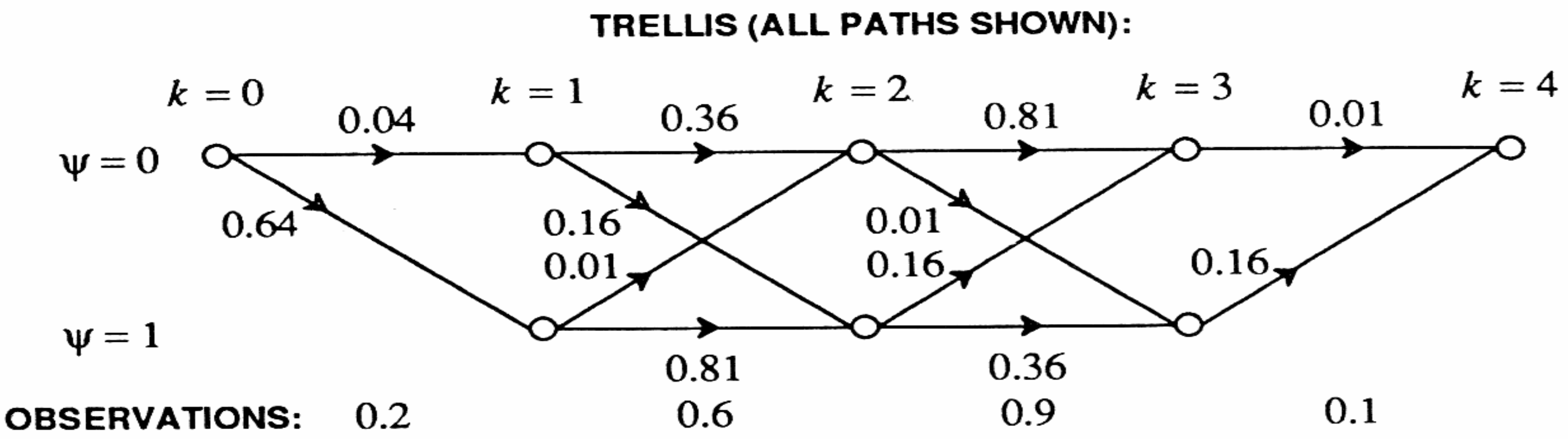
- Observation sequence {0.2, 0.6, 0.9, 0.1}
- Impulse response of channel

$$h_k = \delta_k + 0.5\delta_{k-1}$$

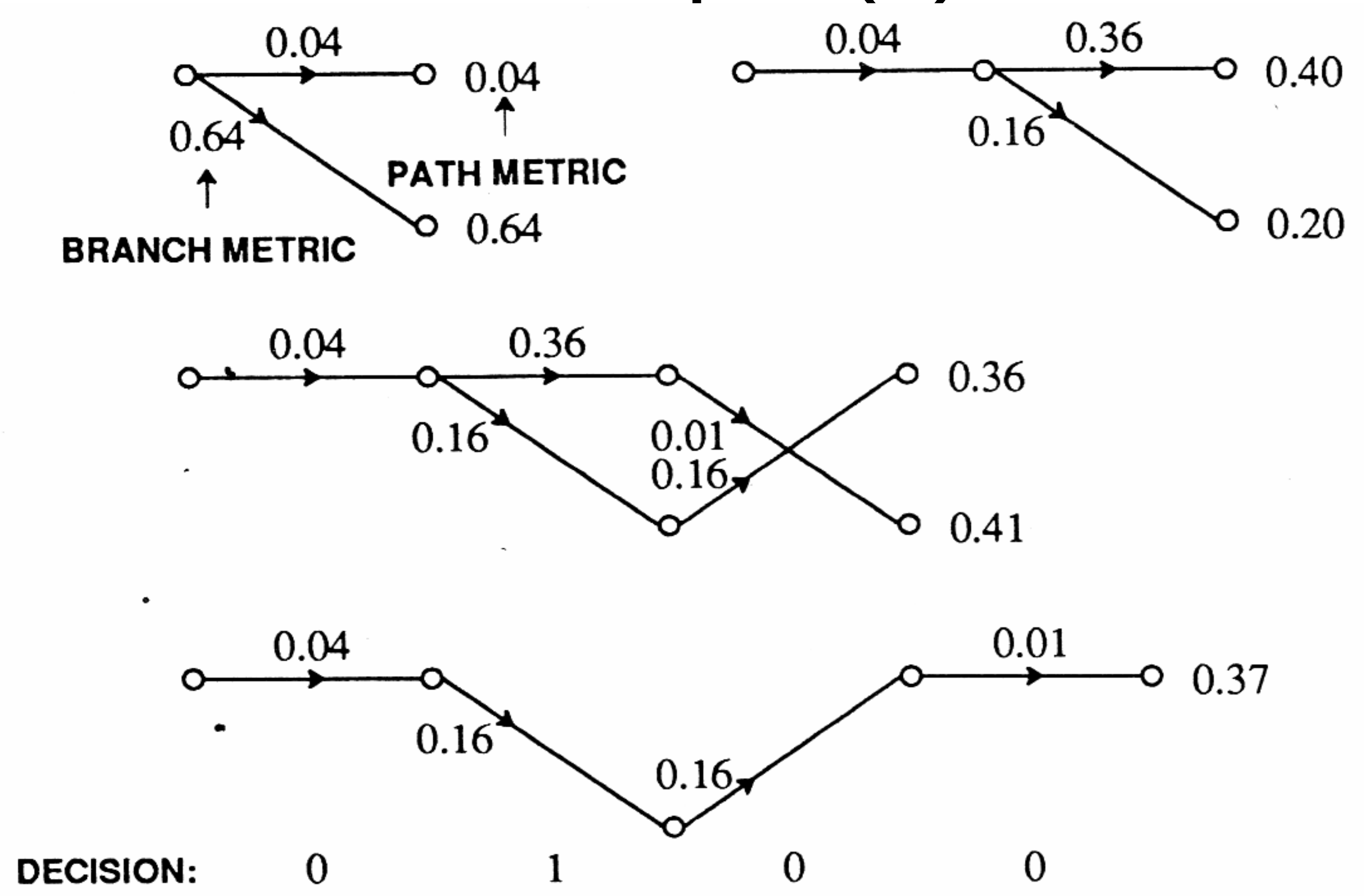
- AWGN
- State transitions



The Viterbi Algorithm Example (2)



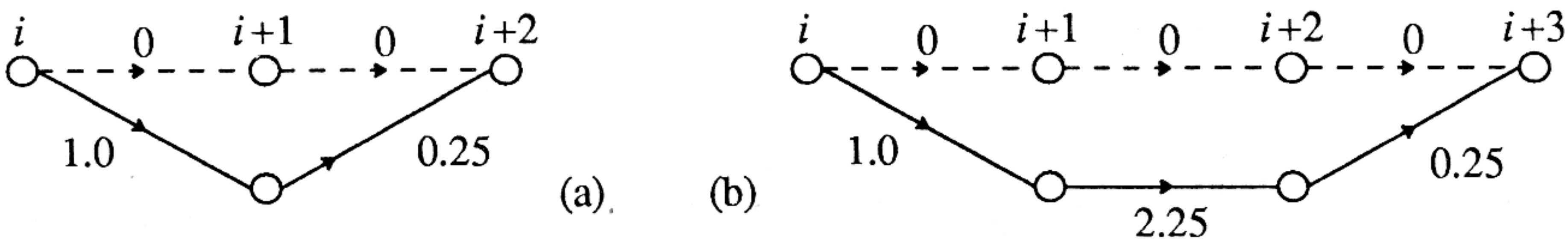
The Viterbi Algorithm Example (3)



Error Probability Calculation

- Error Event
- Detection Error
- Upper Bound of Detection Error
- Lower Bound of Detection Error
- Symbol Error Probability

Error Event



- (a) length 1,
  metric from real sequence $\sqrt{1.25}$
- (b) length 2,
  metric from real sequence $\sqrt{3.5}$

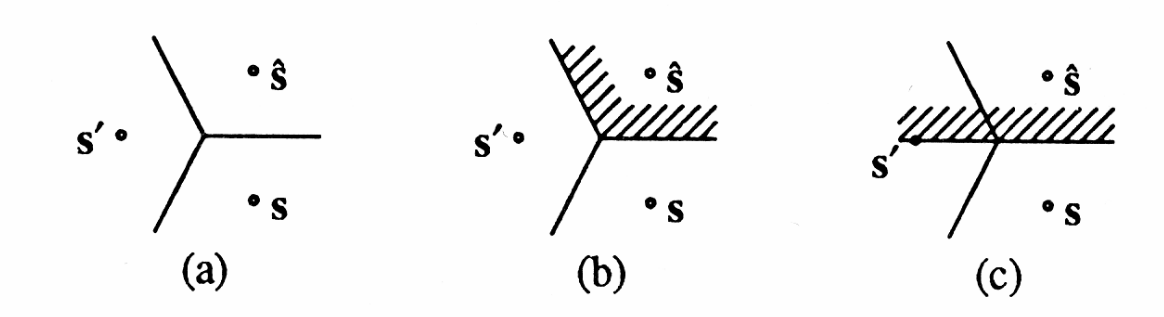
Detection Error

$$\Pr[\text{detection error}] = \sum_{e \varepsilon E} \Pr[e] w(e)$$

$$\Pr[e] = \Pr\left[\Psi\right]\Pr\left[\hat{\Psi} \middle| \Psi\right]$$

- w(e) … total number of detection errors in error event e
- Pr[e] depends on real path and chosen path ➔ estimate $\Pr\left[\hat{\Psi} \middle| \Psi\right]$

Upper Bound of Detection Error (1)



$$\Pr\left[\hat{\Psi} \mid \Psi\right] \leq Q\left(d(\hat{\Psi}, \Psi)/2\sigma\right)$$

- Q … cumulative probability distribution
- d …Euclidian distance of real an chosen path

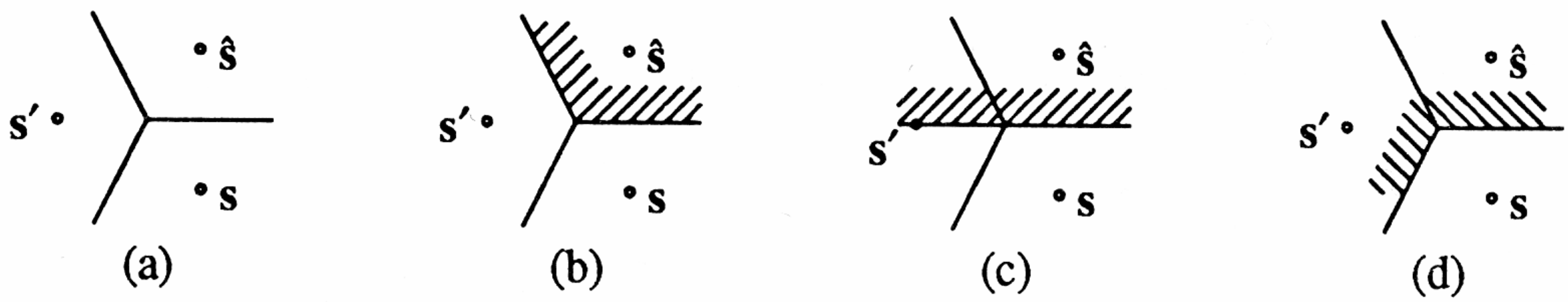
Upper Bound of Detection Error (2)

$$\Pr[\text{detection error}] \leq \sum_{e \varepsilon E} w(e) \Pr[\Psi] Q\left(d_{\min} / 2\sigma\right) + \text{other terms}$$

- Only terms of minimal distance
- Others decay exponentially
- Approaches $RQ(d_{\min} / 2\sigma)$

$$R = \sum_{e \varepsilon B} w(e) \Pr[\Psi]$$

Lower Bound of Detection Error (1)



$$\Pr[\text{detection error}] \geq \sum_{e \varepsilon E} \Pr[e] = \Pr[\text{an error event}]$$

$$\Pr[\text{an error event} \,|\Psi] \geq Q(d_{\min}(\Psi)/2\sigma)$$

Lower Bound of Detection Error (2)

- Using total probability

$$\Pr[\text{detection error}] \geq \sum_{\Psi} \Pr[\Psi] Q(d_{\min}(\Psi)/2\sigma)$$

- Only minimal distance error events

$$\Pr[\text{detection error}] \geq P Q(d_{\min}(\Psi)/2\sigma)$$

$$P = \sum_{\Psi \varepsilon A} \Pr[\Psi]$$

Symbol Error Probability (1)

- Upper and lower bound together

$$PQ(d_{\min}/2\sigma) \leq \Pr[\text{detection error}] \leq RQ(d_{\min}(\Psi)/2\sigma)$$

- Consider C between P and R

$$\Pr[\text{detection error}] \approx CQ(d_{\min}/2\sigma)$$

Symbol Error Probability (2)

- One detection error, one ore more bit errors
- One input X_k by n source bits

$$\frac{1}{n}\Pr[\text{detection error}] \leq \Pr[\text{bit error}] \leq \Pr[\text{detection error}]$$

$$\Pr[\text{detection error}] \approx \Pr[\text{bit error}]$$