Multi User Detection 2

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Outline

- Optimum Detection Theories
    - jointly optimum, synchronous channel
    - individually optimum, synchronous channel
    - jointly optimum, asynchronous channel
- Performance of the Optimal Detector

Optimum Detection Theory

- Goal:

  Minimum achievable probability of error

  Serves as a baseline of comparison for suboptimum detectors

- Main principels:

  Individually Optimum Detector

    Minimizes probability of error for one user

  Jointly Optimum Detector

    Minimizes probability of error for all users

Optimum Detection Theory

- Example:

  A posteriori probabilities (APP) for two users transferring one bit each:

  $P[(+1,+1)|\{y(t), 0 \leq t \leq T\}] = 0.26$

  $P[(-1,+1)|\{y(t), 0 \leq t \leq T\}] = 0.26$

  $P[(+1,-1)|\{y(t), 0 \leq t \leq T\}] = 0.27$

  $P[(-1,-1)|\{y(t), 0 \leq t \leq T\}] = 0.21$

→ Individually optimum detector: (+1,+1)

→ Jointly optimum detector: (+1, -1)

Optimum Detection Theory

- The single user matched filter is the optimum detector, *if the detector only knows y1.*

- The *optimum* receiver knows all used waveforms, timings and amplitudes of all users and the noise level.

Jointly optimum, synchronous channel

- Two-user synchronous channel:

$$y(t) = A_1 b_1 s_1(t) + A_2 b_2 s_2(t) + \sigma n(t), t \in [0, T]$$

The four possible (b_1,b_2) are equiprobable:

$$x_1 = A_1 s_1 + A_2 S_2$$
$$x_2 = A_1 s_1 - A_2 S_2$$
$$x_3 = -A_1 s_1 + A_2 S_2$$
$$x_4 = -A_1 s_1 - A_2 S_2$$

Jointly optimum, synchronous channel

Demanded is the pair (b_1,b_2) that maximizes:

$$f[y(t), 0 \le t \le T|(b_1 b_2)] =$$
$$\exp(-\frac{1}{2\sigma^2}\int_0^T [y(t) - A_1 b_1 s_1(t) - A_2 b_2 s_2(t)]^2 dt)$$

It can be expressed as:

$$f[y(t), 0 \le t \le T|(b_1 b_2)] =$$
$$\exp(-\frac{1}{\sigma^2}\Omega_2(b_1, b_2)) exp(-\frac{A_1^2 + A_2^2}{2\sigma^2}) exp(-\frac{1}{2\sigma^2}\int_0^T y^2(t)dt)$$

Jointly optimum, synchronous channel

→ Maximize Ω_2:

$$\Omega_2(b_1, b_2) = b_1 A_1 y_1 + b_2 A_2 y_2 - b_1 b_2 A_1 A_2 \rho$$

If $\min\{A_1|y_1|, A_2|y_2|\} \ge A_1 A_2|\rho|$

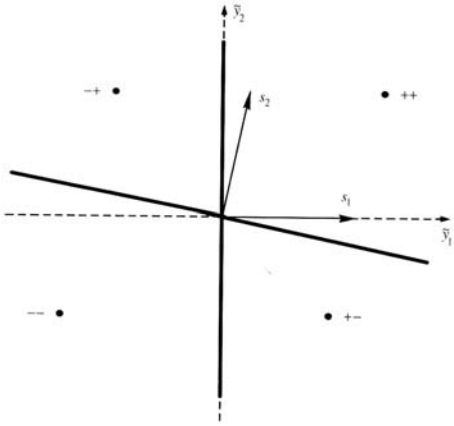
$$\hat{b}_1 = sgn(y_1)$$
$$\hat{b}_2 = sgn(y_2)$$

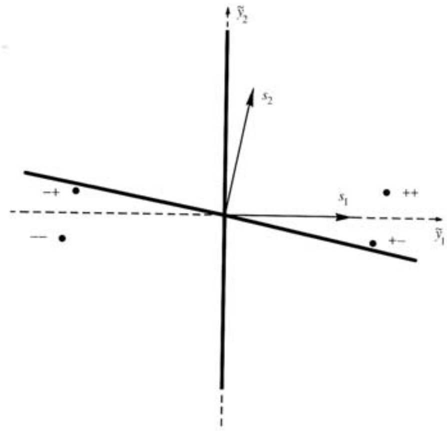
Otherwise

$$\hat{b}_1 = sgn(A_1 y_1 - sgn(\rho) A_2 y_2)$$
$$\hat{b}_2 = sgn(A_2 y_2 - sgn(\rho) A_1 y_1)$$

Jointly optimum, synchronous channel

Decision regions for the single user matched filter detector:

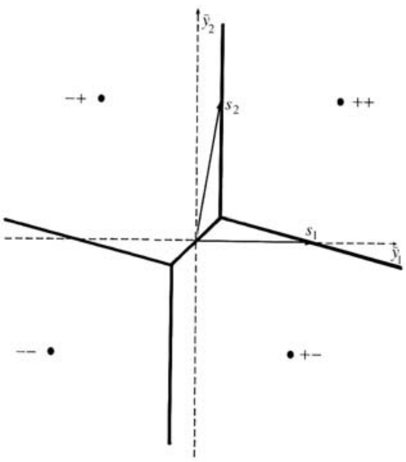


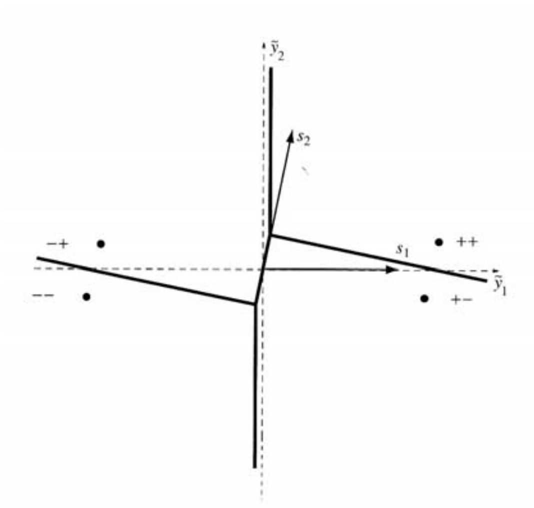


Left: $A_1 = A_2$, right: $A_1 = 6A_2$, Figures taken from [1]

Jointly optimum, synchronous channel

Decision regions for the jointly optimum filter detector:

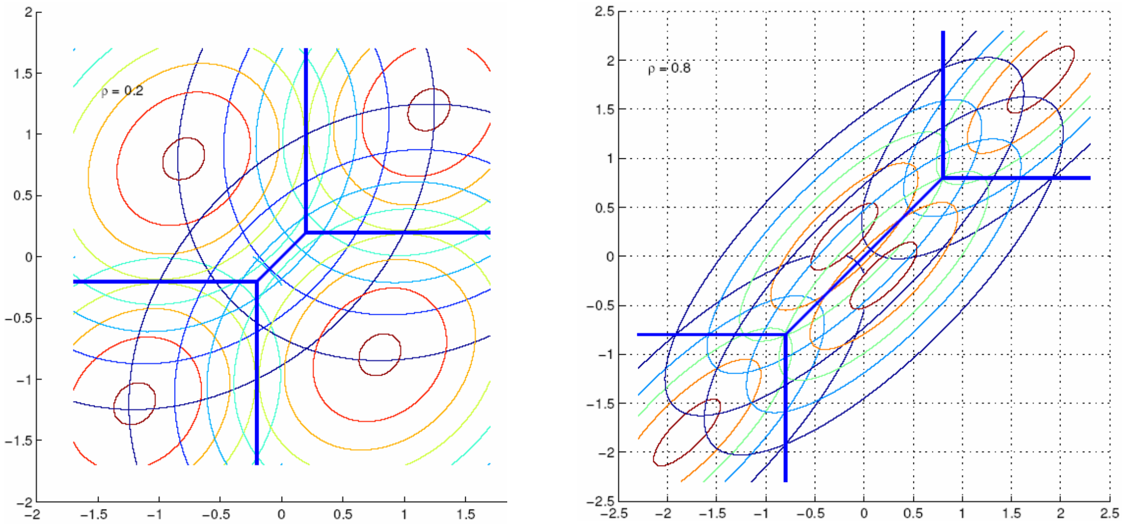




Left: $A_1 = A_2$, right: $A_1 = 6A_2$, $\rho=0.2$, Figures taken from [1]

Jointly optimum, synchronous channel

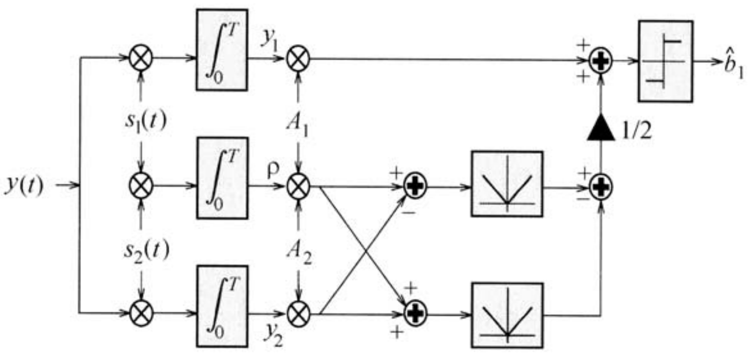
Dependency from Crosscorrelation:



Left: ρ=0.2, right: ρ=0.8, Figures taken from [2]

Jointly optimum, synchronous channel

A possible implementation:



$$\hat{b}_1 = sgn(A_1 y_1 + \frac{1}{2}|A_2 y_2 - A_1 A_2 \rho| - \frac{1}{2}|A_2 y_2 + A_1 A_2 \rho|)$$

$$\hat{b}_2 = sgn(A_2 y_2 + \frac{1}{2}|A_1 y_1 - A_1 A_2 \rho| - \frac{1}{2}|A_1 y_1 + A_1 A_2 \rho|)$$

Figure taken from [1]

Jointly optimum, synchronous channel

K-user channel: $y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t), t \in [0, T]$

$$\mathbf{b} = [b_1, ..., b_k]^T$$

b has to be jointly optimum demodulated and therefore maximize:

$$\exp(-\frac{1}{2\sigma^2} \int_0^T [y(t) - \sum_{k=1}^{K} b_k A_k s_k(t)]^2 dt)$$

Jointly optimum, synchronous channel

As in the Two-User case, this leads to the maximization of a parameter Ω:

$$\Omega(\mathbf{b}) = 2\mathbf{b}^T \mathbf{A}\mathbf{y} - \mathbf{b}^T \mathbf{A}\mathbf{R}\mathbf{A}\mathbf{b}$$

Y are again the matched filter outputs,

A = diag{A_1,…, A_k}

→ time complexity $O(2^K)$, can be reduced when certain restrictions on **R** are imposed.

Individually optimum, synchronous channel

This detector maximizes the symbol-wise APP of b_k: $P(b_k|y(t))$

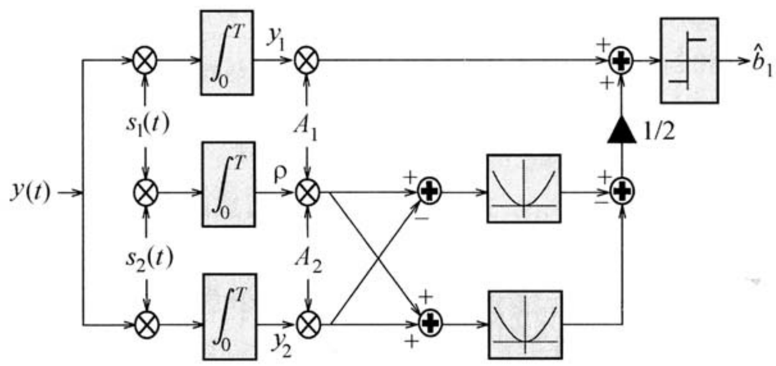
$$P(b_k|y(t)) = \sum_{\mathbf{b}_{\bar{k}}} P(b_k, \mathbf{b}_{\bar{k}}|y(t))$$

The solution to the resulting maximization problem is:

$$\hat{b}_1 = sgn[y_1 - \frac{\sigma^2}{2A_1} ln \frac{cosh(\frac{1}{\sigma^2}(A_2 y_2 + A_1 A_2 \rho))}{cosh(\frac{1}{\sigma^2}(A_2 y_2 - A_1 A_2 \rho))}]$$

Individually optimum, synchronous channel

A possible implementation:



The nonlinearity is: $f_\sigma(x) = \sigma^2 log(cosh(\frac{x}{\sigma^2}))$

Figure taken from [1]

Individually optimum, synchronous channel

Decision regions for the individually optimum detector:

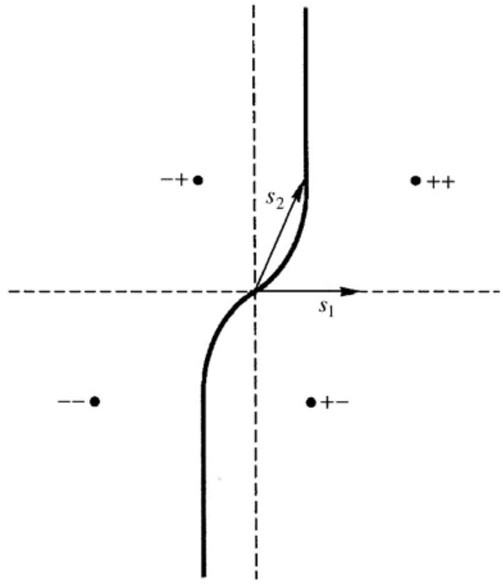


Figure taken from [1]

Individually optimum, synchronous channel

Since $$\lim_{\sigma \to 0} f_\sigma(x) = |x|$$

$$f_\sigma(x) = \sigma^2 log(cosh(\frac{x}{\sigma^2}))$$

$$\hat{b}_1 = sgn[y_1 - \frac{\sigma^2}{2A_1} ln \frac{cosh(\frac{1}{\sigma^2}(A_2 y_2 + A_1 A_2 \rho))}{cosh(\frac{1}{\sigma^2}(A_2 y_2 - A_1 A_2 \rho))}]$$

we see that, with increasing SNR, the individually and the jointly optimum decisions converge.

Individually optimum, synchronous channel

The solutions converge with growing SNR:

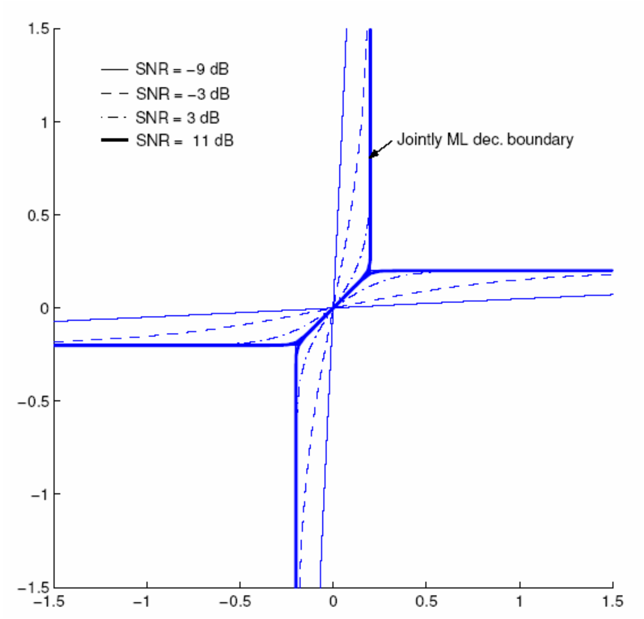


Figure taken from [2]

Jointly vs. Individually optimum detector

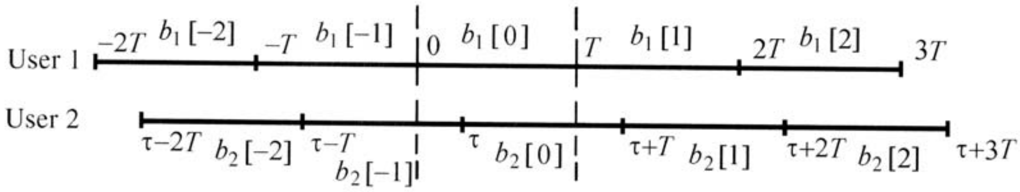
- Jointly optimum decision independent from noise
- Individually optimum detector easier to define, but a closed-form solution for an arbitrary number of users not achievable
- Individually optimum solution converges to the jointly optimum one for high SNR

→ Therefore, the jointly optimum detector is usually referred to as the ‚optimum detector‘

- Hard/Soft Decisions possible for further assessment

Optimum Detector for Asynchronous Channels

In the asynchronous channel, an optimum detector has to observe all the bits of a frame transmitted by every user:



$b_1(0)$ requires knowledge of $b_2(0)$ and $b_2(-1)$ etc.

Figure taken from [1]

Jointly optimum, asynchronous channel

K-user, M-frame (M+1 bits / frame) asynchronous channel can be treated as a K(2M+1)-user synchronous channel.

Three-User-Example:

K=3, M=1 (2 bits transmitted per user)

$\mathbf{b} = [b_1, b_2, b_3, b_4, b_5, b_6]^T$, $b_1=b_1(0)$, $b_4=b_1(1)$, etc.

$\mathbf{A} = \mathrm{diag}(A_1, A_2, \ldots, A_6)$

Jointly optimum, asynchronous channel

Assuming that the time delays between users are less than one symbol inervall, the correlation matrix is:

$$\mathbf{R} = \begin{pmatrix} 1 & \rho_{21} & \rho_{13} & 0 & 0 & 0 \\ \rho_{21} & 1 & \rho_{23} & \rho_{24} & 0 & 0 \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} & \rho_{35} & 0 \\ 0 & \rho_{42} & \rho_{43} & 1 & \rho_{45} & \rho_{46} \\ 0 & 0 & \rho_{53} & \rho_{54} & 1 & \rho_{56} \\ 0 & 0 & 0 & \rho_{64} & \rho_{65} & 1 \end{pmatrix}$$

As stated before, the optimal solution has to maximize

$$\Omega(\mathbf{b}) = 2\mathbf{b}^T\mathbf{Ay} - \mathbf{b}^T\mathbf{ARAb}$$

Jointly optimum, asynchronous channel

Goal: Use of the Viterbi Algorithm to solve the problem

→ $\Omega(\mathbf{b})$ should depend only on a state vector $\mathbf{x}_j$ and b_j

$$\Omega(\mathbf{b}) = \Sigma_{j=1}^{K(M+1)} \lambda_j(\mathbf{x}_j, b_j)$$

$$\Omega(\mathbf{b}) = 2\Sigma_{j=1}^{K(M+1)} A_j b_j y_j - \mathbf{b}^T\mathbf{ARAb}$$

→ Find $\alpha_j(\mathbf{x}_j, b_j)$, so that

$$\lambda_j(\mathbf{x}_j, b_j) = A_j b_j y_j - \alpha_j(\mathbf{x}_j, b_j)$$

Jointly optimum, asynchronous channel

A_j can be found as:

$$\alpha_j(\mathbf{x}_j, b_j) = \mathbf{b}^T \mathbf{ARAb}$$

$$\alpha_j(\mathbf{x}_j, b_j) = A_j b_j (A_j b_j + 2 \overrightarrow{\rho}_j^T \Lambda_j \mathbf{x}_j)$$

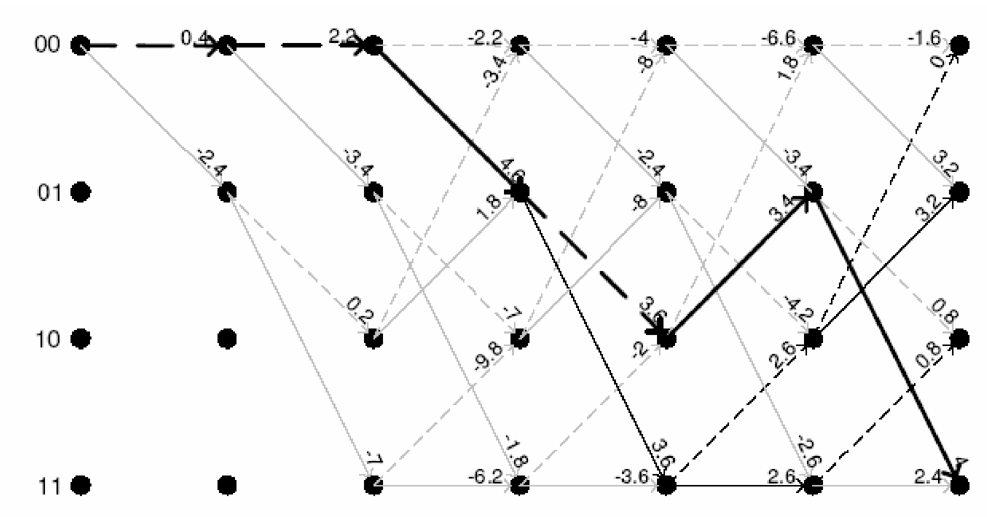
$$\overrightarrow{\rho}_j = [\rho_{j-2,j}, \rho_{j-1,j}]^T$$

$$\Lambda_j = diag(A_{j-2}, A_{j-1})$$

$$\mathbf{x}_j = [b_{j-2}, b_{j-1}]^T$$

Jointly optimum, asynchronous channel

The Trellis diagram (for a numerical example):



Grey: rejected branches
Solid: input bit 1, Dashed: input bit 0

Figure taken from [2]

References

[1] ‚Multiuser Detection‘, Sergio Verdu, Cambridge University Press

[2] ‚Optimum Multiuser Detection‘, T. J. Lim, Notes for ECE1530S 2002/2003, Part III

[3] ‚Optimum Multiuser Detection is Tractable for Synchronous CDMA Systems Using M-Sequences‘, S. Ulukus, R. D. Yates, IEEE Communications Letter, Vol. 2, No. 4, April 1998