Advanced Signal Processing

Fundamentals of Detection
Theory 1 by
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Fundamentals of Detection Theory

- Problem Statements
- Mathematical formulation and techniques
- Decision Theory

Detection Theory in Signal Processing

- Radar
- Communications
- Speech
- Sonar
- Image Processing
- Biomedicine
- Control
- Seismology

Radar Systems

- Electromagnetic pulse
- Reflection on a large moving object

Detection	Decision
noise	nothing
echo in noise	aircraft

Communication Systems

- BPSK System

Detection	Decision
Zero in noise	Zero sent
One in noise	One sent

Speech Systems

- Which Digit from 0 to 9 was spoken?

Detection	Decision
“Zero” in noise	“Zero” spoken
“One” in noise	“One” spoken

Central Problem

- Determining the Function of the Data
- Mapping it into a decision

→ Decision-making based on data

→ statistical hypothesis testing

The Detection Problem

- Signal present
- Only noise present

➔ **Binary Hypothesis Testing Problem**

➔ **Multiple Hypothesis Testing Problem**

The Mathematical Detection Problem

Detection of a DC level of amplitude A = 1 embedded in Gaussian noise ω[n] with variance σ^2. We have to decide between two Hypotheses:

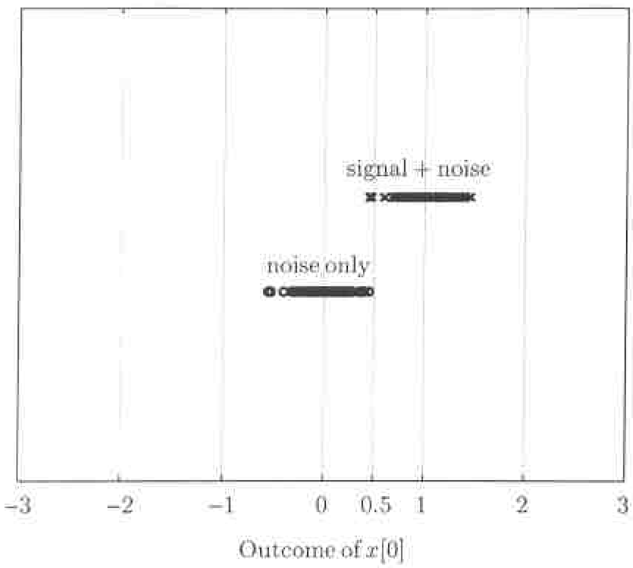
- H0 : x[0] = ω[0] (noise only)
- H1 : x[0] = 1 + ω[0] (signal in noise)

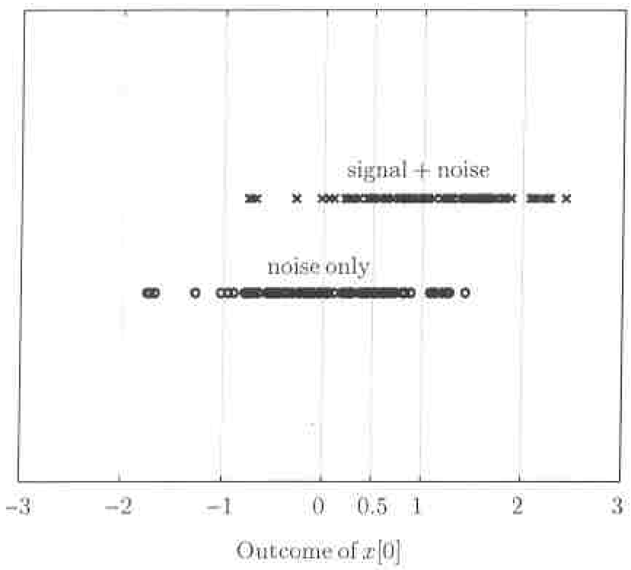
Since noise is assumed to be zero mean:

- x[0] > ½ (Signal is present)
- x[0] < ½ (Only noise is present)

Repeat the Experiment

Observe x[0] for 100 realisations





$\sigma^2 = 0{,}05$ $\sigma^2 = 0{,}5$

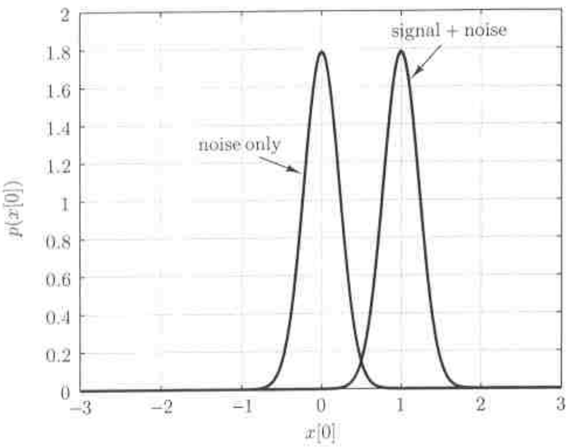
The PDF

- The Probability Density Function of noise

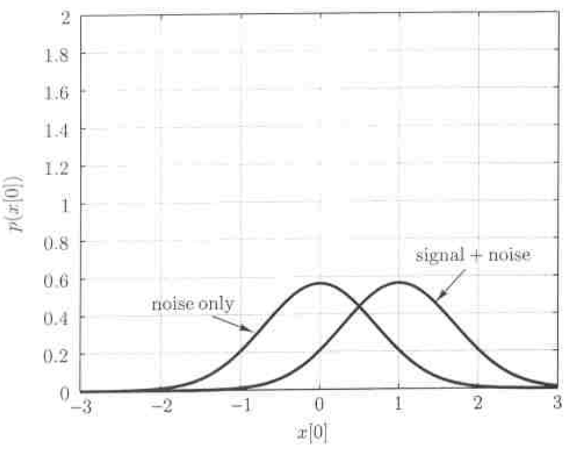
$$p(\omega[0]) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\omega^2[0]\right)$$

The performance of each detector will depend upon how different the PDFs of x[0] are under each Hypothesis.

PDFs for x[0] for signal present or absent



$\sigma^2 = 0{,}05$



$\sigma^2 = 0{,}5$

Choosing between H0 and H1

- H0 : x[0] = ω[0] (Noise only Hypothesis)
- H1 : x[0] = 1 + ω[0] (Signal present Hypothesis)

$$p(x[0];H0)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}\omega^2[0]\right)$$

$$p(x[0];H1)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x[0]-1)^2\right)$$

The Gaussian PDF

- Normal PDF

$$p(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

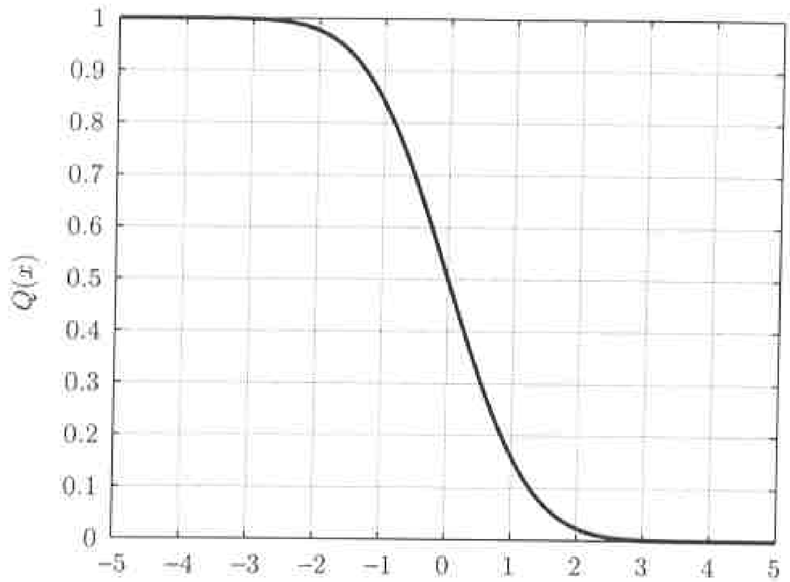
- μ is the mean
- σ^2 is the Variance
- It is denoted $N(\mu,\sigma^2)$

Special:

- Cumulative Distribution Function → $\Phi(x)$
- Right Tail Probability or Complementary Cumulative Distribution Function → $Q(x) = 1 - \Phi(x)$

Right Tail Probability

- The right-tail Probability is the probability of exceeding a given value



CHI-SQUARED PDF (Central)

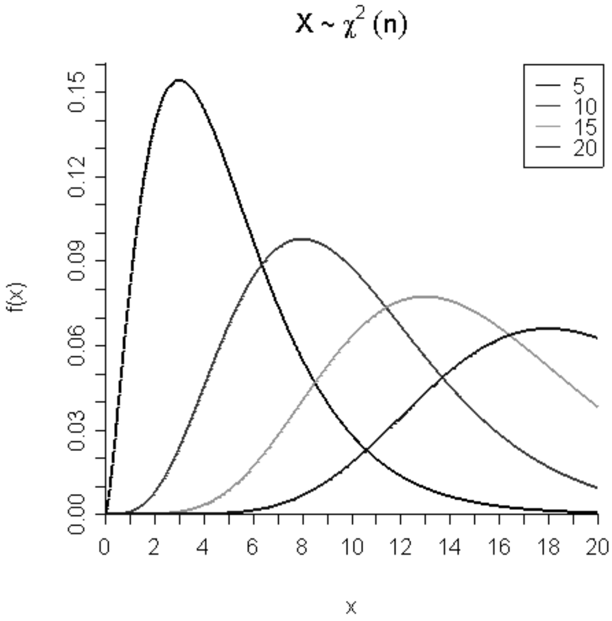
- The chi-squared PDF is denoted by χ^2_n and is defined as

$$f(x) = \frac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \exp\left(-\frac{x}{2}\right) \quad 0 < x < \infty$$

- n is an Integer with n>= 1 and is called freedom
- It becomes Gaussian if n becomes large
- It arises as the PDF where x=sum(χ^2) for i=1 to n if χ ~ N(0,1) and χ's are independent and identically distributed IID

CHI-SQUARED PDF (Central) (2)

- For n=1 the PDF is infinite at x = 0



- E(x)=n
- var(x)=2n

Decision Theory

Simple hypothesis testing Problem

Classical Approach:

- Neyman Pearson theorem
- Bayesian approach based on minimum Bayes risk

Neyman-Pearson Theorem (1)

- Observe a realisation of a random variable whose PDF is either N(0,1) or N(1,1)
- $N(\mu,\sigma^2)$ denotes Gausian PDF with mean μ and variance σ^2
- Only a single observation x[0]
- We have to choose among two competing hypotheses:

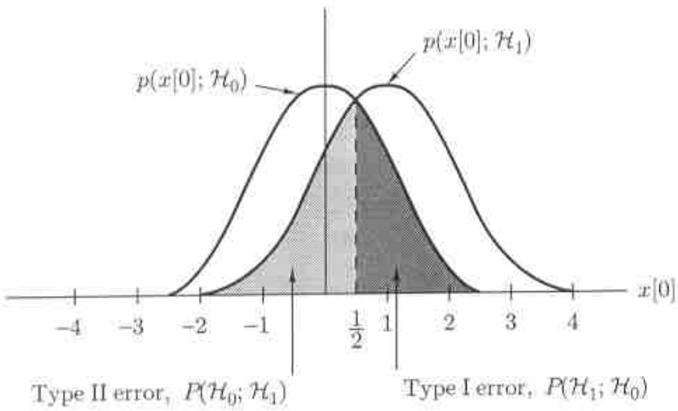
hypotheses:	H0 : μ = 0	(Null H.)
	H1 : μ = 1	(Alternative H.)

➔ Binary hypothesis test

Error Types

- Decide H1 if x[0] > ½ ➔ threshold

- ERROR Type I → decide H1 but H0 is true
- ERROR Type II → decide H0 but H1 is true



These Errors are unavoidable!

Probabilities

- Hypotheses → H0 : X[0] = ω[0]
  → H1 : X[0] = s[0] + ω[0]
- S[0]=1 and ω[0] ~ N(0,1)

- P(H1;H0) Probability of False Alarm → P_{FA}
- P(H1;H1) Probability of Detection → P_D

We wish to maximize P_D subject to the constraint $P_{FA} = \alpha$

Neyman-Pearson Theorem (2)

- Theorem:
  To maximize for a given decide H1 if

$$L(x) = \frac{p(x;H1)}{p(x;H0)} > \gamma$$

where the threshold is found from

$$P_{FA} = \int_{\{x;L(x)>\gamma\}} p(x;H0)dx = \alpha$$

L(x) is termed the *likelihood ratio*

Minimum Probability of Error

- In digital communications a “0” or “1” is equal likely
- Now we have: → H0 (“0” sent)
  → H1 (“1” sent)

P(H0) = P(H1) = ½ where P(H0) and P(H1) are the prior probabilities of the respective Hypothesis. This type of approach is the Bayesian approach.

Minimum Probability of Error (2)

- Probability of error P_e

P_e = P_r{decide H0,H1 true} + P_r{decide H1,H0 true
= P(H0|H1)P(H1) + P(H1|H0)P(H0)

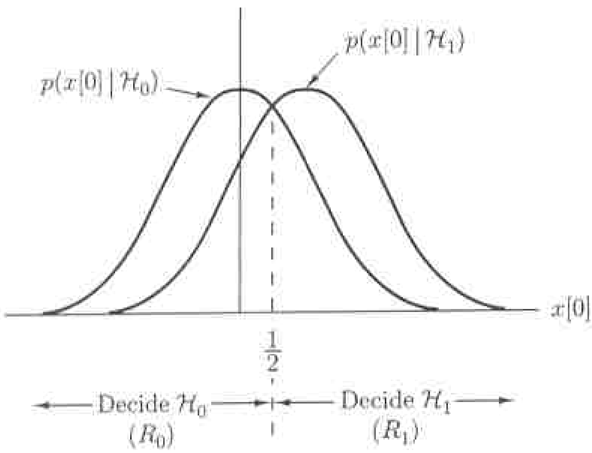
- Minimizing the error probability

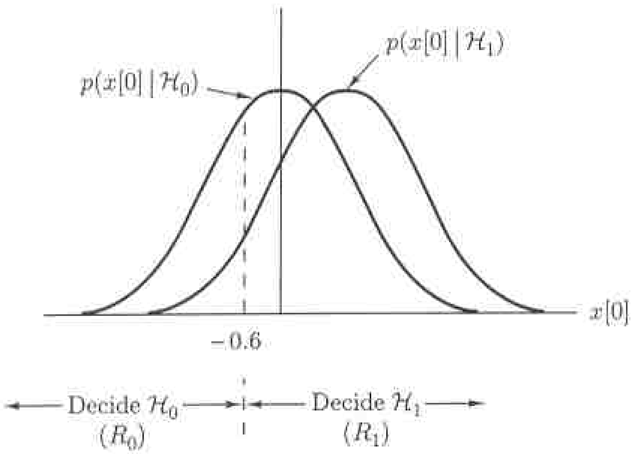
Decide H1 if: P(H1|x) > P(H0|x)

$$\frac{p(x \mid H1)}{p(x \mid H0)} > \frac{P(H0)}{P(H1)} = \gamma$$

MAP Detector

- minimizes Pe for any prior probability
- Effect of prior probability on decision regions
  1. P(H0) = P(H1) = ½
  2. P(H0) = 1/4 , P(H1) = ¾





Cross-Reference of Statistical Terms for Binary Hypothesis Testing

Statisticians	Engineers
Test statistic (T(x) and threshold (γ))	Detector
Null hypothesis (H0)	Noise only hypothesis
Alternative Hypothesis (H1)	Signal + noise hypothesis
Critical Region	Signal present decision Region
Type I error	False Alarm (F_A)
Type II error	Miss (M)
Level of significance (α)	Probability of FA (P_{FA})
Probability of Type II error (β)	Probability of miss (P_M)
Power of test (1- β)	Probability of detection (P_D)