Presentation

Signal space representations of communication signals, optimal detection and error probability

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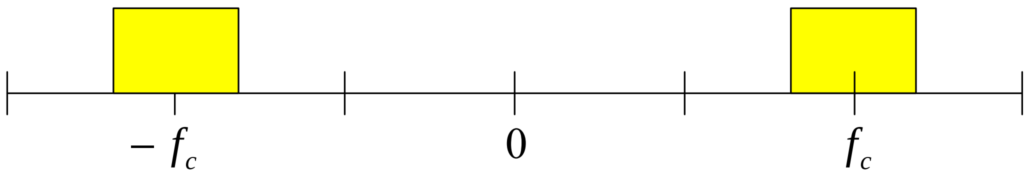
Februari 2005

Outline

- Representation of bandpass signal
- Discrete Time equivalent model
- Signal space representations
- Optimal detection on AWGN Channel
- Symbol Error Probability on AWGN Channel

bandpass signal

- Bandpass Signal
  - Real valued signal   S(f) ⇔ $S^*(-f)$
  - finite bandwidth B ⇔ infinite time span
  - f_c denotes center frequency



  - Negative Frequencies contain no Additional Info

Bandpass to Baseband

$$\hat{s}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}\,d\tau$$

- Two step procedure:

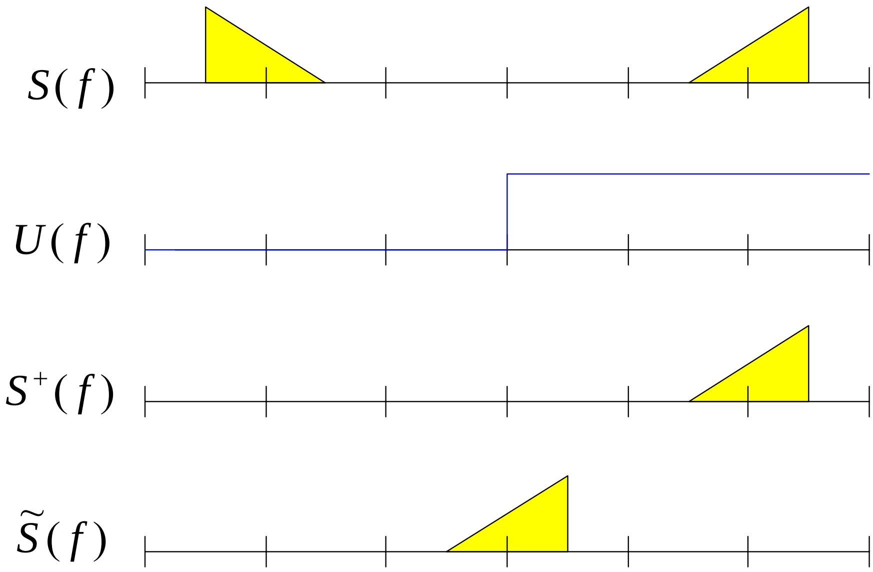
$$S^+(f) = 2U(f)S(f) \quad \Leftrightarrow \quad s^+(t) = \tfrac{1}{2}s(t) + j\tfrac{1}{2}\hat{s}(t)$$

$$\tilde{S}(f) = S^+(f - f_c) \quad \Leftrightarrow \quad \tilde{s}(t) = s^+(t)\exp(-j2\pi f_c)$$

- Characteristics:
  - Complex valued signal
  - No information loss, truely equivalent
- Reconstruction:

$$s(t) = 2\,\mathrm{Re}\left[\tilde{s}(t)\exp(j2\pi f_c)\right]$$

Graphical impression



Discrete Time equivalent model

- Ideal Sampling process with period T

  $s[k] \mathrel{\hat{=}} s(kT)$

- Reconstruction Process

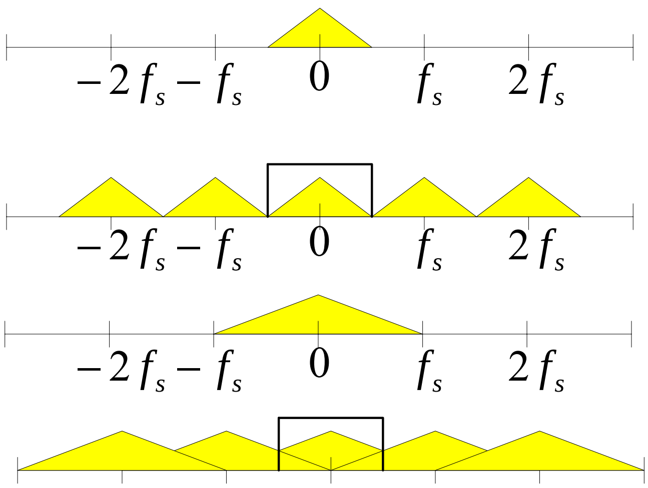
  $\hat{s}(t) \mathrel{\hat{=}} \sum_k s[k]\delta(kT) = s(t)\sum_k \delta(kT) \qquad \hat{S}(f) \mathrel{\hat{=}} \sum_k S\left(f - \frac{k}{T}\right)$

  - results is spectral copies
  - Use ideal filter to get rid of them

  $s(t) = \sum_k s[k]\operatorname{sinc}(f_s t - k)$

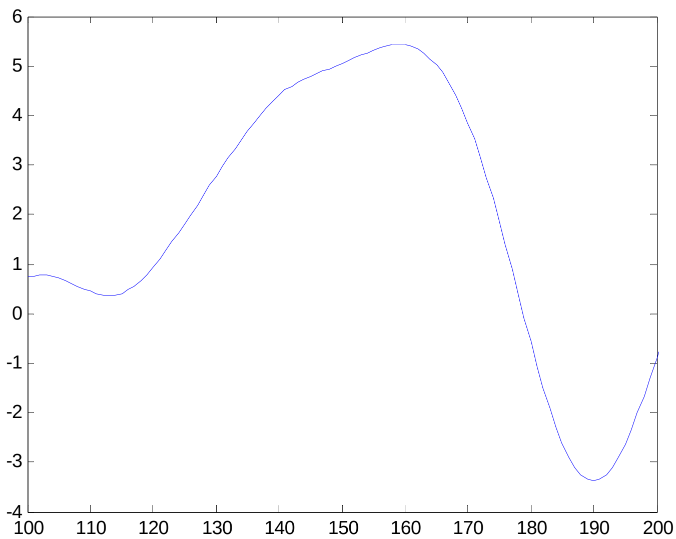
  $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \qquad H(f) = \begin{cases} 1 & |f| < \frac{1}{2} f_s \\ 0 & otherwise \end{cases}$

Graphical Proof of Nyquist

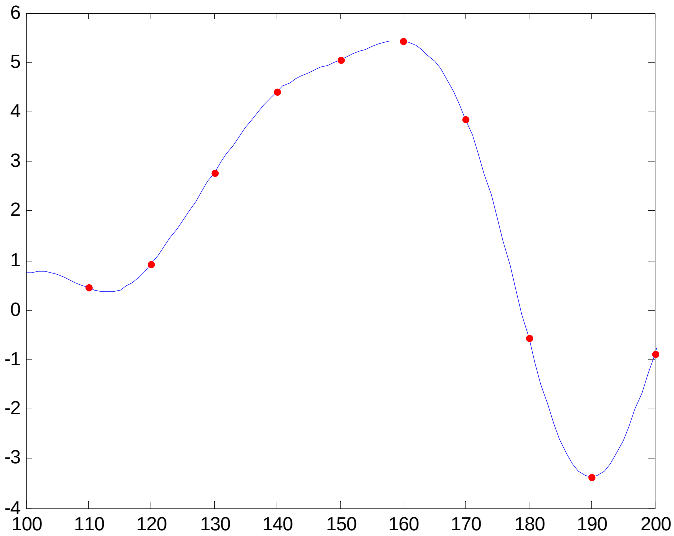


Aliasing

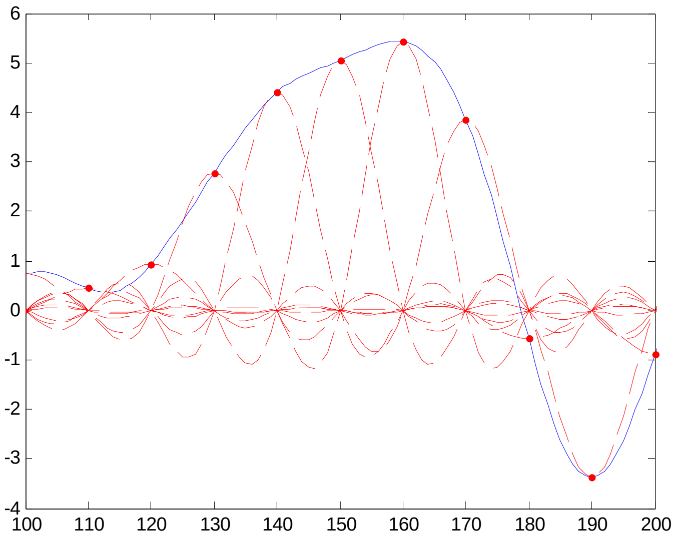
Time Domain Sampling Procedure



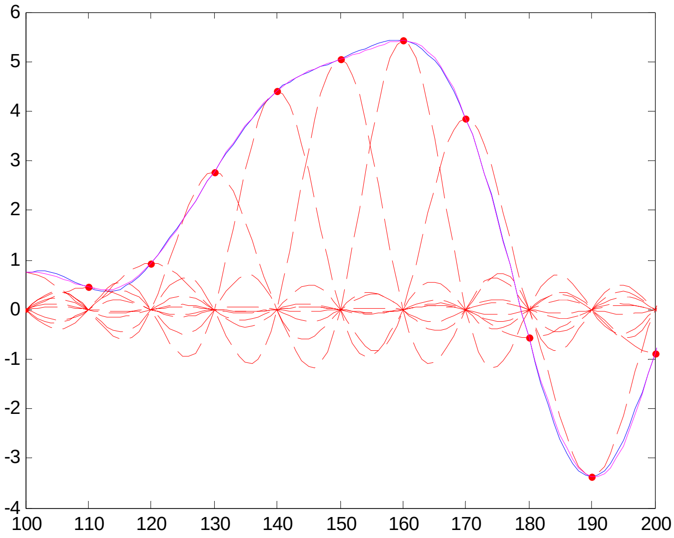
Time Domain Sampling Procedure (2)



Time Domain Sampling Procedure (3)

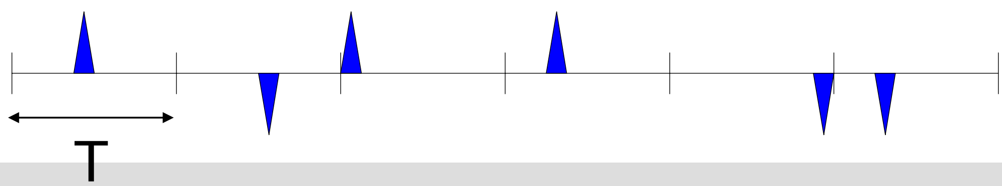


Time Domain Sampling Procedure (4)



General Communication signal

- General Shape: $s(t) \hat{=} \sum_k \sqrt{E_{b_k}}\, w_{b_k}(t-kT)$
  - b_k is value of k-th symbol
  - b_k element of $\{1,2,..,N_s\}$
- Waveform $w_{b_k}(t)$ has:
  - Unit energy for simplicity
  - Duration <T, Bandwidth <B
  - for every possible b_k



Signal Space Representation

- Signal/vector space is a set of vectors together with two operators, addition of vector and multiplication by a scalar
- Define a set of 2BT real-valued orthonormal functions $f_1(t), f_2(t), \ldots, f_{BT}(t)$ spanning the 2BT-dimensional space

IMST

Signal Space Representation

- Each waveform can be described by a vector containing 2BT elements

$$\mathbf{w} = \int_{-\infty}^{\infty} w(t)\, \boldsymbol{f}(t)\, dt$$

$$\boldsymbol{f}(t) = \left[f_1(t) \quad f_2(t) \quad \ldots \quad f_{2BT}(t) \right]^T$$

- Some properties:

$$\int_{-\infty}^{\infty} |w(t)|^2 dt = \|\mathbf{w}\|^2 \qquad \int_{-\infty}^{\infty} v(t) w(t) dt = \langle \mathbf{v}, \mathbf{w} \rangle$$

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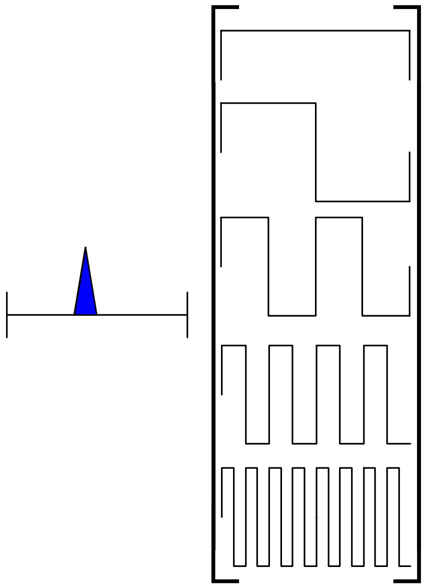
Signal Space Representation

- Example

$$\boldsymbol{w} = \int_0^T$$



$$dt$$

Optimal detection on AWGN Channel

- Received signal vector

$$\boldsymbol{r} = \boldsymbol{w}_{b_k} + \boldsymbol{n}$$

- Noise is not bandwidth limited, but a proper receiver „looks“ only in the waveform space
- Noise elements are i.i.d. Gaussian RV
- Receiver must make decision on transmitted waveform based on $\boldsymbol{r}$
- Maximum likelihood (ML) receiver
  - Given AWGN case and equal likely symbols,
  - Maximum Likelihood = Minimum Distance

Optimal detection on AWGN Channel

- Symbol Error Prob. (SEP) is minimized if:

$$\hat{b} = \arg\min_{k \in B} \left(\left\| \boldsymbol{r} - \boldsymbol{w}_k \right\|^2 \right)$$

- Distance can be written out to:

$$\left\| \boldsymbol{r} - \boldsymbol{w}_k \right\|^2 = \boxed{\left\| \boldsymbol{r} \right\|^2} - 2\langle \boldsymbol{r}, \boldsymbol{w}_k \rangle + \left\| \boldsymbol{w}_k \right\|^2$$

The same for all k

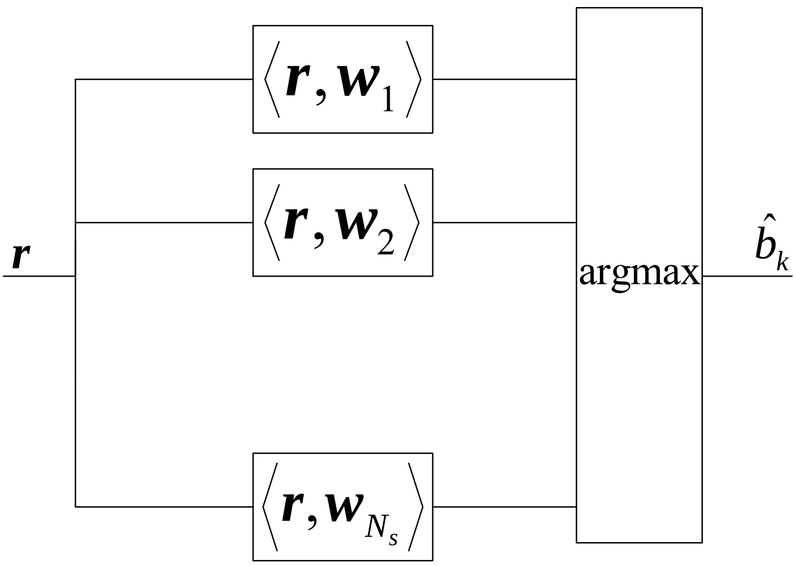
- In case of equal energy symbols

$$\hat{b} = \arg\max_{k \in B} \left(\boxed{\langle \boldsymbol{r}, \boldsymbol{w}_k \rangle} \right)$$

Output of b-th Matched Filter

Optimal detection on AWGN Channel

- Signal Space Matched Filter receiver



Note: for equal Energy Symbols

Optimal detection on AWGN Channel

- 2-th order Bi-orthogonal (BPSK)
  - BPSK is 1 dimensional Modulation $\langle \mathbf{r}, \mathbf{w}_1 \rangle = -\langle \mathbf{r}, \mathbf{w}_2 \rangle$
- 4-th order Bi-orthogonal (QPSK)
  - QPSK is 2 dimensional $\langle \mathbf{r}, \mathbf{w}_1 \rangle = -\langle \mathbf{r}, \mathbf{w}_2 \rangle$ $\langle \mathbf{r}, \mathbf{w}_3 \rangle = -\langle \mathbf{r}, \mathbf{w}_4 \rangle$
  - Noise at the two MF outputs is independent $\langle \mathbf{w}_{1,2}, \mathbf{w}_{3,4} \rangle = 0$
- The waveform space is subspace of 2BT space
- N_s-th order Bi-orthogonal modulation
  - Biorthogonal modulation is $N_s/2$ dimensional
  - Maximum order is $\frac{1}{2}M < 2BT$

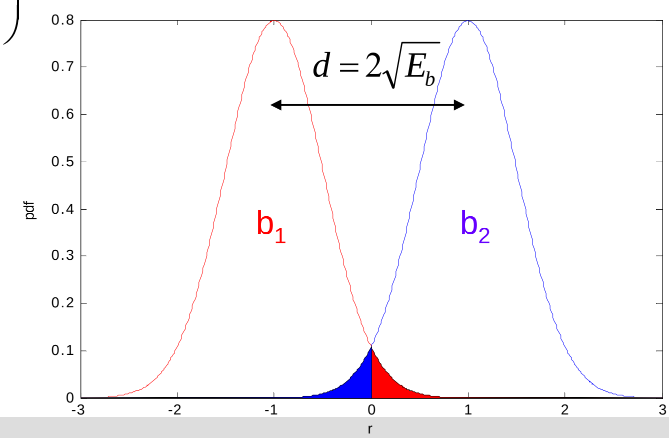
Symbol Error Probability on AWGN channel

- BPSK case

$$P(e) = \tfrac{1}{2} P(\langle \mathbf{r}, \mathbf{w}_2 \rangle > \langle \mathbf{r}, \mathbf{w}_1 \rangle \mid b_k = 1) + \tfrac{1}{2} P(\langle \mathbf{r}, \mathbf{w}_1 \rangle > \langle \mathbf{r}, \mathbf{w}_2 \rangle \mid b_k = 2)$$

$$P(\langle \mathbf{r}, \mathbf{w}_2 \rangle > \langle \mathbf{r}, \mathbf{w}_1 \rangle \mid b_k = b_1) = Q\left(\sqrt{\frac{d_{21}^2}{2N_0}}\right)$$

$$P(e) = Q\left(\sqrt{\frac{d_{21}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



Symbol Error Probability on AWGN channel

- Closed form derivation of higher order modulation often impossible
- Solution: Union-bound

$$P(e) \le \sum_{b \in B} P(e \mid b_k = b) P(b_k = b) = \frac{1}{M}$$

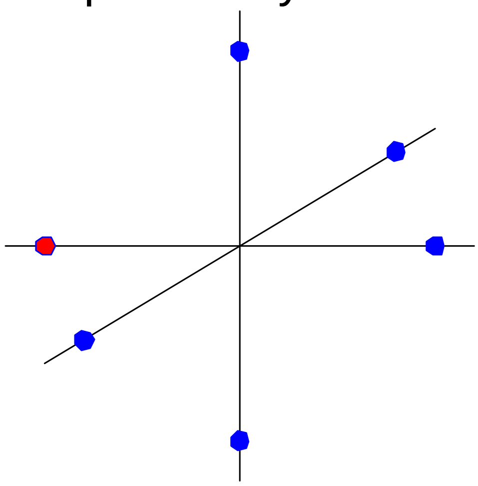
$$P(e \mid b_k = b) \le \sum_{d \in B, d \ne b} P(\langle \mathbf{r}, \mathbf{w}_d \rangle > \langle \mathbf{r}, \mathbf{w}_b \rangle \mid b_k = b) = Q\left(\sqrt{\frac{d_{db}^2}{2N_0}}\right)$$

$$P(e) \le \frac{1}{M} \sum_{b \in B} \sum_{d \in B, d \ne b} Q\left(\sqrt{\frac{d_{db}^2}{2N_0}}\right)$$

Symbol Error Probability on AWGN channel

- 6-th order bi-orthogonal modulation
  - Each symbol has the same error probability
    - Mirror Symmetry

$$P(e) \le \sum_{d \in B, d \ne b} Q\left(\sqrt{\frac{d_{db}^2}{2N_0}}\right)$$

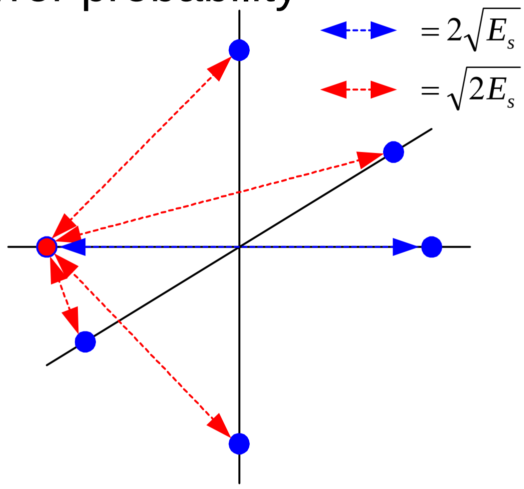


Symbol Error Probability on AWGN channel

- 6-th order bi-orthogonal modulation
  - Each symbol has the same error probability
    - mirror Symmetry

$$P(e) \le \sum_{d \in B, d \ne b} Q\left(\sqrt{\frac{d_{db}^2}{2N_0}}\right)$$

$$P(e) \le 4Q\left(\sqrt{\frac{2E_s}{2N_0}}\right) + Q\left(\sqrt{\frac{4E_s}{2N_0}}\right)$$



Question: 6-th order Bi-orthogonal worse than BPSK???

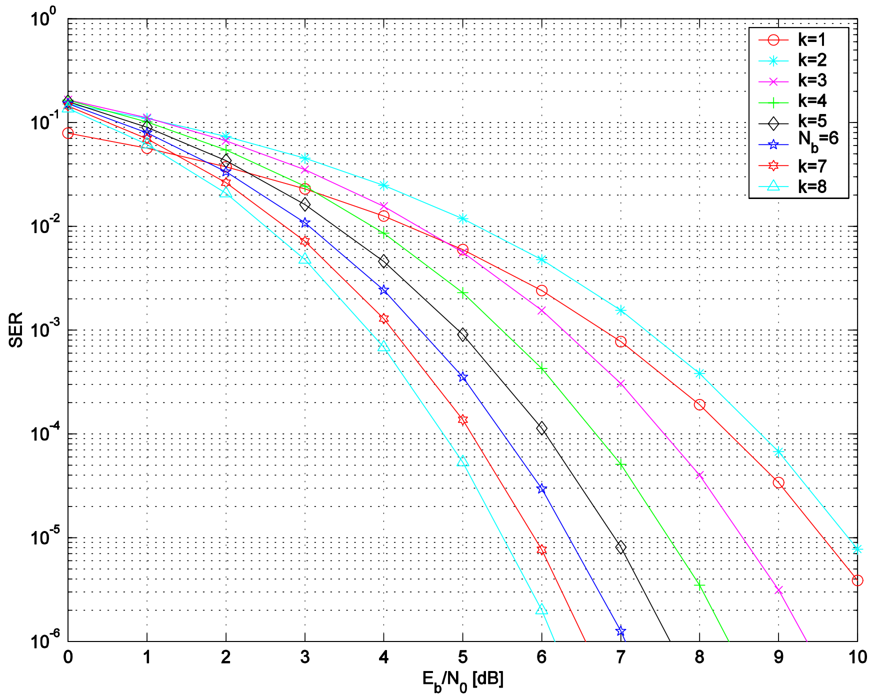
Symbol Error Probability on AWGN channel

- Answer: yes and no
  - Yes: with respect to E_s
  - No: with respect to E_b $\{ E_s = \log_2(6) E_b \}$

$$P(e) \le 4Q\left(\sqrt{\frac{2\log_2(6)E_b}{2N_0}}\right) + Q\left(\sqrt{\frac{4\log_2(6)E_b}{2N_0}}\right)$$

- Second No: Symbol Error is not same as Bit error
  - Depends on bits to symbol mapping
    - Gray Coding
  - See Proakis

SEP of Bi-orthogonal modulation



Note: numerically obtained exact SEP

Any Questions??