Optimum Detection of Deterministic and Random Signals

Advanced Signal Processing 1 SE

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Outline

- **Detection of deterministic signals**
  - Problem statement
  - Replica-correlator
  - Matched filter
  - Generalized matched filter
- **Detection of random signals**
  - Problem statement
  - Energy detector
  - Estimator correlator

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Detection Problem

- *Problem statement:*

$$\begin{array}{lll} H_0: & x[n] = w[n] & n = 0, \cdots, N-1 \\ H_1: & x[n] = s[n] + w[n] & n = 0, \cdots, N-1 \end{array}$$

s[n]: known *deterministic* signal

w[n]: zero-mean, white Gaussian noise (WGN) with variance σ^2

- The detection problem is to distinguish between these two hypotheses

Replica-Correlator

- The Neyman-Pearson (NP) detector decides H_1 if the likelihood ratio exceeds a threshold

$$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathbf{H_1})}{p(\mathbf{x};\mathbf{H_0})} > \gamma$$

- Use the PDF of the data under both hypotheses

$$p(\mathbf{x};\mathbf{H_1}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\mathbf{e}^{-\frac{1}{2\sigma^2}\sum_{\mathbf{n=0}}^{\mathbf{N-1}}(\mathbf{x[n]}-\mathbf{s[n]})^{\mathbf{2}}}$$

$$p(\mathbf{x};\mathbf{H_0}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\mathbf{e}^{-\frac{1}{2\sigma^2}\sum_{\mathbf{n=0}}^{\mathbf{N-1}}\mathbf{x^2[n]}}$$

Replica-Correlator (2)

- Taking the logarithm of both sides does not change the inequality

$$\begin{aligned} l(\mathbf{x}) &= lnL(\mathbf{x}) \\ &= -\frac{1}{2\sigma^2}(\sum_{n=0}^{N-1}(x[n]-s[n])^2 - \sum_{n=0}^{N-1}x^2[n]) > ln\gamma \end{aligned}$$

- We decide H_1 if

$$\frac{1}{\sigma^2}\sum_{n=0}^{N-1}x[n]s[n] - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}s^2[n] > ln\gamma$$

Replica-Correlator (3)

- Incorporating the energy term into the threshold yields

$$T(\mathbf{x}) = \sum_{\mathbf{n=0}}^{\mathbf{N-1}} \mathbf{x[n]s[n]} > \sigma^2 \mathbf{ln}\gamma + \frac{1}{2}\sum_{\mathbf{n=0}}^{\mathbf{N-1}} \mathbf{s^2[n]}$$

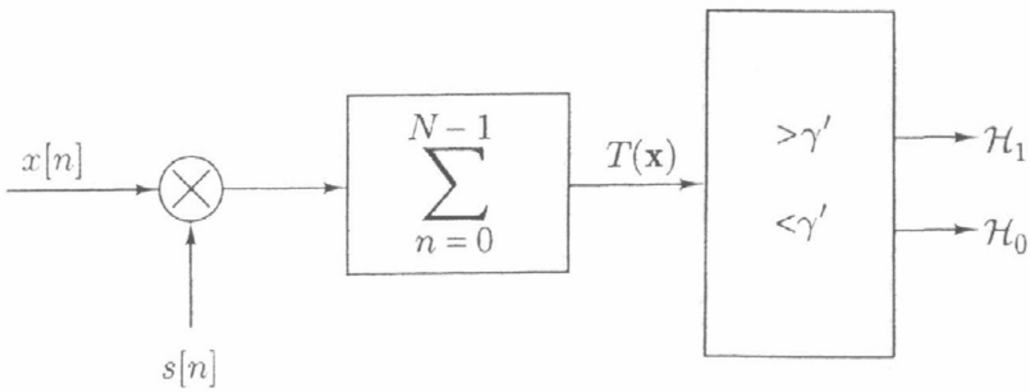
- With the new threshold γ', we decide H_1 if

$$T(\mathbf{x}) = \mathbf{x^T s} = \sum_{\mathbf{n=0}}^{\mathbf{N-1}} \mathbf{x[n]s[n]} > \gamma'$$

- This is the NP-detector for a deterministic signal in WGN (= *Replica-correlator*)

Replica-Correlator (4)

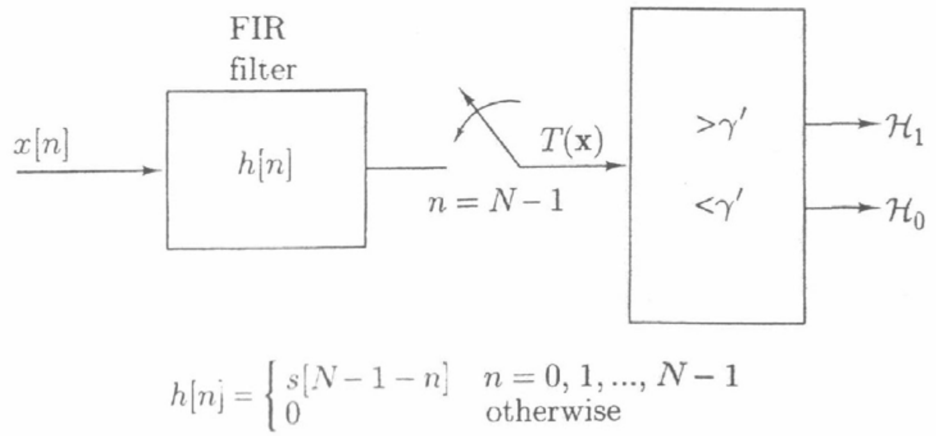
- *Replica-correlator*:



– Received data is *correlated* with signal *replica*

Matched Filter

- Alternative interpretation: *Matched filter*



  – Correlation process as FIR filtering of data

Matched Filter (2)

- Let h[n] be a „flipped around“ version of s[n]

$$h[n] = s[N-1-n] \qquad n = 0, \cdots, N-1$$

- Output of the filter at time n = N-1

$$y[N-1] = \sum_{k=0}^{N-1} s[k]x[k] = T(\mathbf{x})$$

- Impulse reponse is *matched* to the signal
- Signal output is maximal at the sampling time
- Unsuitable for signals with unknown arrival time

Matched Filter (3)

- *Properties of the matched filter*:
  1) Emphasizes bands with higher signal power
     - DTFT of the impulse response

$$H(f) = S^*(f)e^{-j2\pi f(N-1)}$$

  2) Maximizes the SNR at the FIR filter output

$$\eta_{max} = \frac{\mathbf{s}^{\mathbf{T}}\mathbf{s}}{\sigma^2} = \frac{\epsilon}{\sigma^2}$$

     - ε: signal energy

Matched Filter (4)

- *Detection performance*:
  – Scaled test statistic

$$T' \sim \begin{cases} N(0,1) & under\ \ H_0 \\ N(\sqrt{\epsilon/\sigma^2},1) & under\ \ H_1 \end{cases}$$

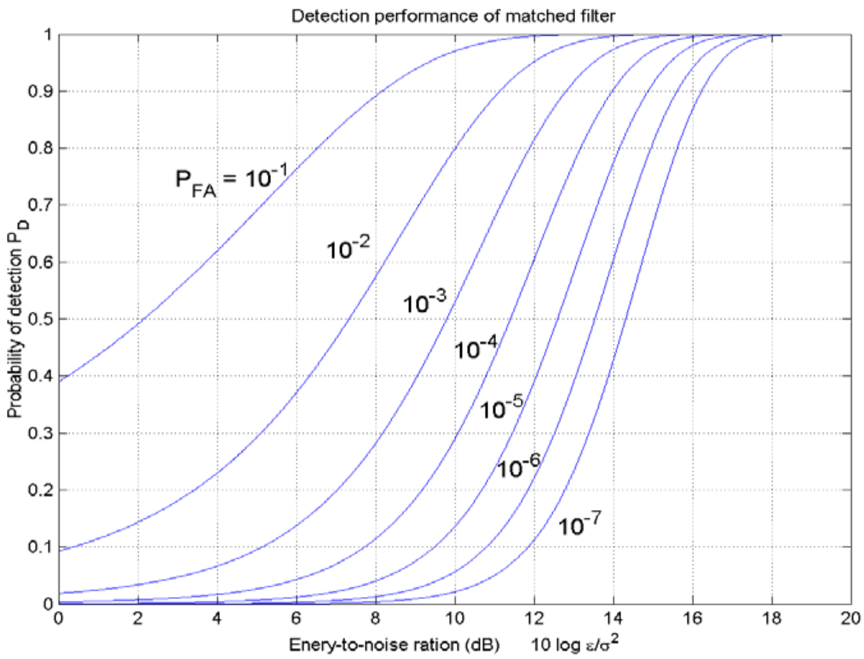
  – Probability of detection

$$\begin{aligned} P_D &= Pr\left\{T > \gamma'; H_1\right\} \\ &= Q(Q^{-1}(P_{FA}) - \sqrt{\epsilon/\sigma^2}) \end{aligned}$$

  – The signal shape does *not* affect the detection performance (in case of WGN)

Matched Filter (6)

- *Probability-of-detection curves*:



Generalized Matched Filter

- Matched filter is an optimal detector for a known signal in WGN
- Assume: noise is *correlated*, i.e. $\mathbf{w} \sim \mathrm{N}(\mathbf{0}, \mathbf{C})$
- Using the same derivation as above, we decide H_1 if

$$T(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}^{\mathrm{T}}\mathbf{s}' > \gamma'$$

- This is the NP-detector for a deterministic signal in colored noise

Generalized Matched Filter (1)

- *Detection performance*:

$$P_D = Q(Q^{-1}(P_{FA}) - \sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}})$$

- Signal shape is relevant (design signal to maximize $\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$ and hence P_D)
- In the WGN case, i.e. $\mathbf{C} = \sigma^2 \mathbf{I}$, we get the same results for T(**x**) and P_D as above

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Detection Problem

- *Problem statement:*

$$H_0: \quad x[n] = w[n] \qquad n = 0, \cdots, N-1$$
$$H_1: \quad x[n] = s[n] + w[n] \qquad n = 0, \cdots, N-1$$

s[n]: zero-mean Gaussian *random* process with known covariance

w[n]: WGN with known variance σ^2; indep. of s[n]

- We decide H_1 if the likelihood ratio exceeds γ

$$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathbf{H_1})}{p(\mathbf{x};\mathbf{H_0})} > \gamma$$

Energy Detector

- First, model signal s[n] as WGN with variance σ_S^2

$$\mathbf{x} \sim \begin{cases} N(\mathbf{0}, \sigma^2\mathbf{I}) & under \;\; H_0 \\ N(\mathbf{0}, (\sigma_S^2 + \sigma^2)\mathbf{I}) & under \;\; H_1 \end{cases}$$

- We decide H_1 if

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = \sum_{n=0}^{N-1} x^2[n] > \gamma'$$

- Computes the *energy* in the received data

Energy Detector (2)

- *Detection performance*:
  - Since the statistic is the sum of the squares of N i.i.d. Gaussian random variables, it's PDF is c*hi-squared*

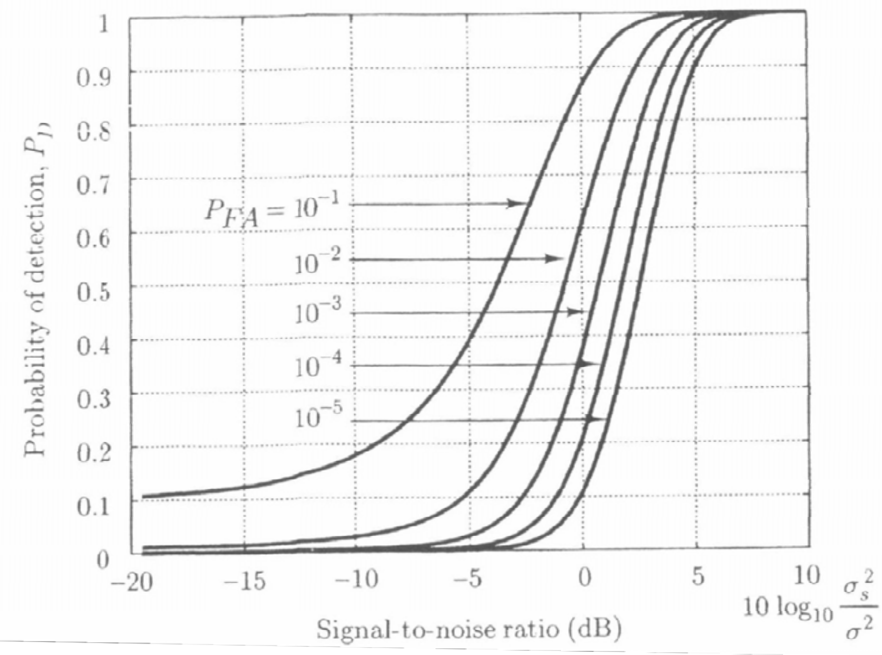
$$\frac{T(\mathbf{x})}{\sigma^2} \sim \chi_N^2 \quad under H_0$$
$$\frac{T(\mathbf{x})}{\sigma_S^2 + \sigma^2} \sim \chi_N^2 \quad under H_1$$

- Probability of detection:

$$P_D = Q_{\chi_N^2}\left(\frac{\gamma'}{\sigma_S^2+\sigma^2}\right) = Q_{\chi_N^2}\left(\frac{\gamma''}{\sigma_S^2/\sigma^2+1}\right)$$

Energy Detector (3)

- *Probability-of-detection curves*:



Estimator-Correlator

- Second, generalize the energy detector to signals with arbitrary covariance matrices $\mathbf{C_S}$

$$\mathbf{x} \sim \begin{cases} N(\mathbf{0}, \sigma^2\mathbf{I}) & under\ H_0 \\ N(\mathbf{0}, \mathbf{C_S} + \sigma^2\mathbf{I}) & under\ H_1 \end{cases}$$

- Decide H_1 if

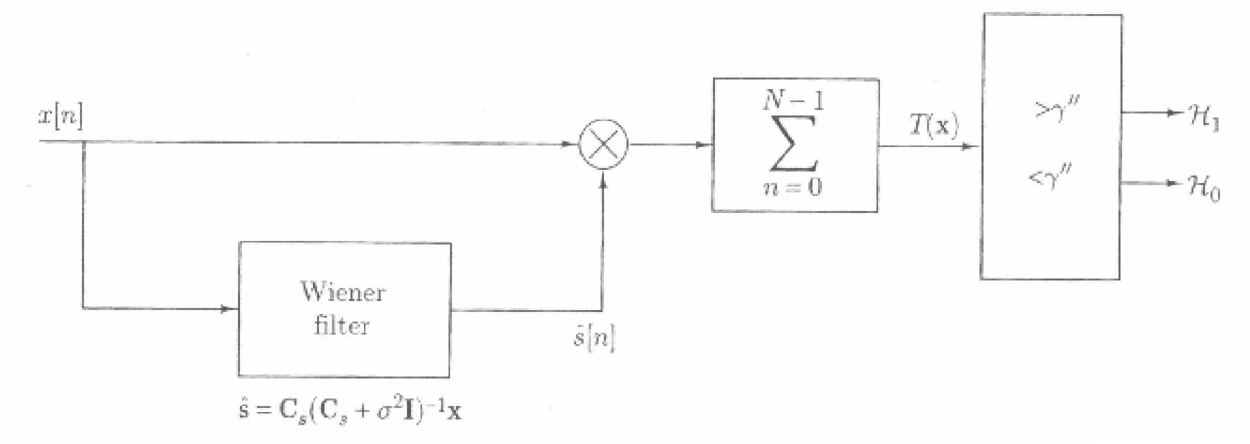
$$T(\mathbf{x}) = \mathbf{x^T\hat{s}} = \sum_{\mathbf{n=0}}^{\mathbf{N-1}} \mathbf{x[n]\hat{s}[n]} > \gamma''$$

where

$$\hat{\mathbf{s}} = \mathbf{C_S}(\mathbf{C_S} + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

Estimator-Correlator (2)

- *Estimator-correlator*:



- The received data is *correlated* with an *estimate* of the signal (compare to replica-correlator)
- $\hat{s}[n]$ is a Wiener filter estimator of the signal

Summary

- **Deterministic signals**
  - The *replica-correlator* is the NP detector (i.e. optimal) for a known signal in WGN
  - The *matched filter* is another implementation
  - The *generalized matched filter* is the NP detector for a signal in correlated noise
- **Random signals**
  - The *energy detector* is the NP detector for a zero mean, white Gaussian signal in WGN
  - The *estimator-correlator is a* generalization of the energy detector to signals with arbitrary covariance matrices