Analysis of Even-Order Terms in Memoryless and Quasi-Memoryless Polynomial Baseband Models

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***Abstract*—Behavioral modeling of nonlinear passband systems like radio frequency power amplifiers is mainly based on polynomial baseband models. Motivated by the convolution property of the Fourier transform applied to passband signals, it is common practice to include only odd-order terms in these models. Experimental results show, however, that significant improvements can be achieved by also including even-order terms. In this paper, the fundamental relationship of even-order terms in polynomial passband and baseband models is analyzed, providing a theoretical foundation for the improved modeling accuracy of polynomial baseband models with even-order terms.**

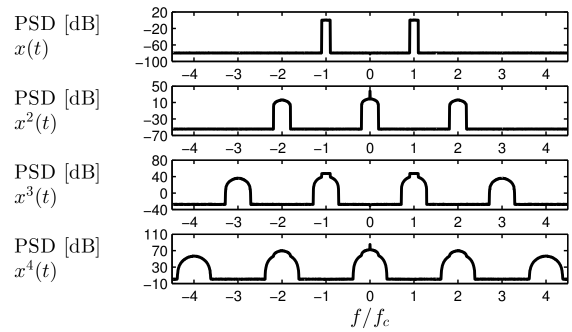

Fig. 1. Power spectral densities (PSDs) of a passband signal raised to powers. Multiplication in time-domain translates to convolution in frequency-domain.

I. INTRODUCTION

Radio frequency power amplifiers are essential building blocks in wireless communication systems. To achieve sufficient linearity, they must be operated at a certain backoff from their maximum power rating. This, however, lowers their efficiency and leads to a trade-off between efficiency and linearity [1]. An effective technique to improve the linearity-efficiency trade-off is digital predistortion [2] where the distortion generated by the power amplifier is precompensated in the digital baseband processing before upconversion [3].

As it is essential for digital predistortion to find low-complexity, but accurate, nonlinear baseband models, much research effort has been spent on this topic [4]. A very common class of nonlinear baseband models relies on polynomial approximation, where in most cases only odd-order terms are considered [5]. The restriction to odd-order terms can be traced back to the classical derivation of the baseband Volterra series [6], but it can also be motivated rather intuitively by the convolution property of the Fourier transform applied to passband signals. From the power spectra in Fig. 1 it can be seen that even powers of a passband signal $x(t)$ generate an output at even multiples of the carrier frequency f_c and odd powers generate an output at odd multiples of the carrier frequency. Since a baseband model only describes the input-output relation within the first spectral zone, given by $f \in \pm[f_\mathrm{c} - f_\mathrm{c}/2, f_\mathrm{c} + f_\mathrm{c}/2]$, the terms $x^p(t)$ with even p of a polynomial passband model have no effect on the equivalent baseband model. However, it is important to note that this does not imply that even-order terms should be excluded from polynomial baseband models. In fact it was shown by experiments that by including even-order terms in quasi-memoryless baseband models [7] and Hammerstein baseband models [8], the power amplifier modeling accuracy, as well as the predistorter performance can be improved. In [9], a new type of baseband Volterra series was proposed which also includes even-order terms and features better modeling accuracy than the conventional baseband Volterra series. In [10], it was shown that even-order terms in memoryless baseband models can explain measured behavior of intermodulation products which can not be explained by odd-order terms only.

Despite these examples for the relevance of even-order terms in polynomial baseband models, the analytical foundations for improved modeling accuracy have not yet been investigated in full detail. In the present paper, such an investigation is presented based on the Chebyshev transform and its inversion [11], which relate general memoryless passband models to their corresponding baseband models. By applying these transformations to polynomial models, explicit transformation pairs are derived which show how even-order terms in memoryless and quasi-memoryless polynomial baseband models relate to their corresponding passband models. Based on these transformation pairs, reasons for improved modeling accuracy, an explanation of the spectral characteristics, and a new interpretation of even-order terms are presented.

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II. Passband-Baseband Transformation

A. Memoryless Nonlinearity

A memoryless nonlinearity in passband is characterized by an instantaneous signal transfer function $f : \mathbb{R} \mapsto \mathbb{R}$ which maps a real input signal $x(t)$ to a real output signal $y(t)$ by

$$y(t) = f(x(t)). \tag{1}$$

The equivalent baseband system is formed by adding an upconverter at the input and a zonal filter and a downconverter at the output. The complex baseband input signal

$$\tilde{x}(t) = a(t)e^{j\varphi(t)} \tag{2}$$

with $a(t)$ being the amplitude modulation and $\varphi(t)$ being the phase modulation is upconverted to the angular carrier frequency ω_c which gives the real passband input signal

$$x(t) = a(t)\cos\left(\omega_\mathrm{c} t + \varphi(t)\right). \tag{3}$$

After going through the memoryless nonlinearity given by (1), the output signal is still periodic in $\theta(t) = \omega_\mathrm{c} t + \varphi(t)$ and can therefore be expanded into the Fourier series [11]

$$y(t) = \frac{1}{2} f_0\left(a(t)\right) + \sum_{k=1}^{\infty} f_k\left(a(t)\right)\cos\left(k\theta(t)\right), \tag{4}$$

where the Fourier coefficients $f_k(\cdot)$ are functions[1] of the input amplitude $a(t)$, given by the integral

$$f_k\left(a\right) = \frac{2}{\pi}\int_0^{\pi} f\left(a\cos(\theta)\right)\cos\left(k\theta\right)d\theta. \tag{5}$$

The transformation in (5) is called Chebyshev transform [11] and relates the instantaneous signal transfer function $f(x)$ to the k-th harmonic envelope transfer function $f_k(a)$. To derive the equivalent baseband model, the passband output signal given by (4) is limited to the first spectral zone by removing all harmonics except the first one, leading to

$$y_1(t) = f_1\left(a(t)\right)\cos\left(\omega_\mathrm{c} t + \varphi(t)\right). \tag{6}$$

After downconversion of (6) and renaming $f_1(\cdot)$ into $\tilde{f}(\cdot)$, which simplifies notation, the baseband output signal is

$$\tilde{y}(t) = \tilde{f}\left(a(t)\right)e^{j\varphi(t)}. \tag{7}$$

To sum up, the transformation from the passband model in (1) to the baseband model in (7) is accomplished by evaluation of the first-order Chebyshev transform, which is given by

$$\tilde{f}\left(a\right) = \frac{2}{\pi}\int_0^{\pi} f\left(a\cos(\theta)\right)\cos\left(\theta\right)d\theta. \tag{8}$$

[1]The same technique was used in [12] for the analytical description of pulse-width modulation. The Fourier series expansion is applied on the time-variable and the amplitude-dependence is modeled by the Fourier coefficients.

B. Memoryless Polynomial

A memoryless polynomial passband model approximates the instantaneous signal transfer function $f(\cdot)$ by

$$y(t) = \sum_{p=1}^{P} \alpha_p x^p(t), \tag{9}$$

where α_p are real-valued coefficients and P is the polynomial order. To derive the equivalent baseband model, (9) is transformed to an envelope nonlinearity by (8) and the result is inserted into (7). This is simplified by noting that (8) is linear in $f(\cdot)$ and therefore the passband polynomial basis functions

$$f_{\text{basis},p}(x) = x^p \tag{10}$$

can be transformed individually. The baseband basis functions are obtained by substituting $f(\cdot) \to (\cdot)^p$ in (8), leading to

$$\tilde{f}_{\text{basis},p}\left(a\right) = \frac{2}{\pi}\int_0^{\pi} a^p\cos^p(\theta)\cos\left(\theta\right)d\theta, \tag{11}$$

pulling out a^p and combining the cosine terms, giving

$$\tilde{f}_{\text{basis},p}\left(a\right) = \left[\frac{2}{\pi}\int_0^{\pi}\cos^{p+1}(\theta)d\theta\right]a^p \tag{12}$$

and evaluating the integral, which is zero for even p and a constant λ_p for odd p, resulting in the baseband basis functions

$$\tilde{f}_{\text{basis},p}\left(a\right) = \begin{cases}\lambda_p a^p & p \text{ is odd}\\ 0 & p \text{ is even}\end{cases} \tag{13}$$

with the scaling factors λ_p for odd p given by

$$\lambda_p = \frac{1}{2^{p-1}}\binom{p}{\frac{p-1}{2}}. \tag{14}$$

Recombining the baseband basis functions in (13) with the coefficients α_p to a polynomial and inserting it into (7) gives

$$\tilde{y}(t) = \left[\sum_{\substack{p=1\\p\text{ is odd}}}^{P}\alpha_p\lambda_p a^p(t)\right]e^{j\varphi(t)}. \tag{15}$$

By defining real-valued baseband coefficients $\tilde{\alpha}_p = \alpha_p\lambda_p$ and moving the phase term into the sum, one arrives at the memoryless baseband polynomial model [13] given by

$$\tilde{y}(t) = \sum_{\substack{p=1\\p\text{ is odd}}}^{P}\tilde{\alpha}_p\tilde{x}(t)|\tilde{x}(t)|^{p-1}. \tag{16}$$

Although this derivation shows that (16) only includes odd-order terms, it does not mean that the inclusion of even-order terms in (16) will have no effect in passband. To analyze this effect, a generalization of (16) with both odd and even-order terms will be transformed from baseband to passband.

III. Baseband-Passband Transformation

A. Memoryless Nonlinearity

A memoryless baseband model is transformed to passband by writing it in the form of (7) and applying the inverse first-order Chebyshev transform on the envelope nonlinearity $\tilde{f}(\cdot)$. This inverse transform [11] is given by

$$f(x) = \frac{1}{2}\int_0^{\pi/2} \left[\tilde{f}\left(x\cos(\theta)\right) + \tilde{f}'\left(x\cos(\theta)\right) x\cos(\theta)\right] d\theta \\ + \text{any even-symmetric function of } x \quad (17)$$

which requires the envelope nonlinearity $\tilde{f}(\cdot)$, as well as its first derivative $\tilde{f}'(\cdot)$. The inverse transform is not unique, since the even-symmetric part of $f(x)$ does not create any output at odd harmonics of the carrier frequency and therefore the baseband model cannot contain any information on this part of $f(x)$. In the following, this ambiguity is resolved by setting the arbitrary, even-symmetric part of $f(x)$ to zero.

B. Memoryless Polynomial

To transform (16) with included even-order terms to passband, the phase term is pulled out, leading to

$$\tilde{y}(t) = \left[\sum_{p=1}^{P} \tilde{\alpha}_p a(t)|a(t)|^{p-1}\right] e^{j\varphi(t)} \quad (18)$$

and by comparison of (18) with (7) one can see that $\tilde{f}(\cdot)$ is given by the term in square brackets which gives

$$\tilde{f}(a(t)) = \sum_{p=1}^{P} \tilde{\alpha}_p a(t)|a(t)|^{p-1}. \quad (19)$$

Since the transform in (17) is linear in $\tilde{f}(\cdot)$ and $\tilde{f}'(\cdot)$, it is again possible to transform the basis functions individually. The basis functions of (19) and their first derivatives are

$$\tilde{f}_{\text{basis},p}(a) = a|a|^{p-1}, \quad (20)$$
$$\tilde{f}'_{\text{basis},p}(a) = p|a|^{p-1}. \quad (21)$$

For the transformation to passband, (20) and (21) are inserted into (17) by applying the substitutions

$$\tilde{f}(\cdot) \rightarrow (\cdot)|(\cdot)|^{p-1}, \quad (22)$$
$$\tilde{f}'(\cdot) \rightarrow p|(\cdot)|^{p-1}, \quad (23)$$

leading to the integral

$$f_{\text{basis},p}(x) = \frac{1}{2}\int_0^{\pi/2} x\cos(\theta)|x|^{p-1}|\cos(\theta)|^{p-1} \\ + p|x|^{p-1}|\cos(\theta)|^{p-1} x\cos(\theta)\, d\theta \quad (24)$$

which is simplified by pulling out $x|x|^{p-1}$ and noting that for the given integration range $\cos(\theta) = |\cos(\theta)|$, resulting in

$$f_{\text{basis},p}(x) = \left[\frac{p+1}{2}\int_0^{\pi/2} \cos^p(\theta) d\theta\right] x|x|^{p-1}. \quad (25)$$

Evaluating the integral in (25) to a constant, one arrives at the passband basis functions given by

$$f_{\text{basis},p}(x) = \frac{1}{\lambda_p} x|x|^{p-1}, \quad (26)$$

with the scaling factor λ_p given by

$$\lambda_p = \frac{1}{2^{p-1}} \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+3}{2}\right)}. \quad (27)$$

After recombining the passband basis functions in (26) with the coefficients $\tilde{\alpha}_p = \lambda_p \alpha_p$ to a polynomial, the scaling factor λ_p cancels out, resulting in the passband model

$$y(t) = \sum_{p=1}^{P} \alpha_p x(t)|x(t)|^{p-1}. \quad (28)$$

IV. Quasi-Memoryless Model

The results that have been derived for the memoryless models can easily be extended to quasi-memoryless models, by noting that a quasi-memoryless model can be described by a quadrature system of two memoryless models [14]. The quasi-memoryless passband model first separates the input signal $x(t)$ and its Hilbert transform $\hat{x}(t)$, which are given by

$$x(t) = a(t)\cos(\omega_c t + \varphi(t)), \quad (29)$$
$$\hat{x}(t) = a(t)\sin(\omega_c t + \varphi(t)) \quad (30)$$

and then individually distorts the two components by

$$y(t) = f(x(t)) - g(\hat{x}(t)) \quad (31)$$

with $f(\cdot)$ and $g(\cdot)$ being memoryless passband nonlinearities. Since the components of the quadrature system are orthogonal, the analysis of Sections II and III is still valid for the in-phase component and can be applied similarly to the quadrature-component. The quasi-memoryless extension of the conventional memoryless passband polynomial in (9) is given by

$$y(t) = \sum_{p=1}^{P} \alpha_p x^p(t) - \sum_{p=1}^{P} \beta_p \hat{x}^p(t). \quad (32)$$

After transformation of (32) to baseband one gets

$$\tilde{y}(t) = \sum_{\substack{p=1 \\ \text{p is odd}}}^{P} \tilde{\alpha}_p \tilde{x}(t)|\tilde{x}(t)|^{p-1} + j \sum_{\substack{p=1 \\ \text{p is odd}}}^{P} \tilde{\beta}_p \tilde{x}(t)|\tilde{x}(t)|^{p-1} \quad (33)$$

which can be written with complex coefficients as

$$\tilde{y}(t) = \sum_{\substack{p=1 \\ \text{p is odd}}}^{P} \tilde{\gamma}_p \tilde{x}(t)|\tilde{x}(t)|^{p-1} \quad \text{with} \quad \tilde{\gamma}_p = \tilde{\alpha}_p + j\tilde{\beta}_p. \quad (34)$$

By including even-order terms in baseband and transforming back to passband, one gets the new passband model

$$y(t) = \sum_{p=1}^{P} \alpha_p x(t)|x(t)|^{p-1} - \sum_{p=1}^{P} \beta_p \hat{x}(t)|\hat{x}(t)|^{p-1}. \quad (35)$$

TABLE I

SUMMARY OF PASSBAND-BASEBAND TRANSFORMATION PAIRS OF QUASI-MEMORYLESS POLYNOMIAL MODELS

	Passband Model	Baseband Model	Coefficient Mapping
(a)	$y(t) = \sum_{\substack{p=1 \\ p \text{ is odd}}}^{P} \alpha_p x^p(t) - \sum_{\substack{p=1 \\ p \text{ is odd}}}^{P} \beta_p \hat{x}^p(t)$	$\tilde{y}(t) = \sum_{\substack{p=1 \\ p \text{ is odd}}}^{P} \tilde{\gamma}_p \tilde{x}(t) \lvert\tilde{x}(t)\rvert^{p-1}$	$\tilde{\gamma}_p = \frac{1}{2^{p-1}} \binom{p}{\frac{p-1}{2}} (\alpha_p + j\beta_p)$
(b)	$y(t) = \sum_{p=1}^{P} \alpha_p x(t) \lvert x(t)\rvert^{p-1} - \sum_{p=1}^{P} \beta_p \hat{x}(t) \lvert\hat{x}(t)\rvert^{p-1}$	$\tilde{y}(t) = \sum_{p=1}^{P} \tilde{\gamma}_p \tilde{x}(t) \lvert\tilde{x}(t)\rvert^{p-1}$	$\tilde{\gamma}_p = \frac{1}{2^{p-1}} \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+3}{2}\right)} (\alpha_p + j\beta_p)$

(a) Conventional transformation pair including only odd-order terms. (b) Generalized transformation pair including both even and odd-order terms.

V. DISCUSSION

A. Transformation Pairs

A summary of the derived transformation pairs is given in Table I. The first row contains the conventional one which includes only odd-order terms. The second row contains the new one which includes both even and odd-order terms. The new transformation pair represents a generalization of the conventional one, since it is identical for odd-order terms, i.e., $x|x|^{p-1} = x^p$ for odd p, but it additionally includes even-order terms of the form $x|x|^{p-1} = \text{sign}(x)x^p$ for even p which are not lost during the transformation to baseband. The reason why they remain is, that although they are *even-order terms*, they are also *odd-symmetric functions* and therefore they produce only *odd-order intermodulation products*.

B. Modeling Accuracy

With respect to modeling accuracy, including even-order terms is advantageous, because lower order terms have better numerical characteristics than higher order terms. This gives an advantage for implementation, requiring less bits for numerical representation and it improves least-squares identification by lowering the condition number of the regression matrix.

C. Spectral Characteristics

If the spectral characteristics of baseband polynomial basis functions of the form $\tilde{x}|\tilde{x}|^{p-1}$ are compared, it can be observed that odd-order terms are bandlimited to p times the input signal bandwidth, but even-order terms are not bandlimited. This results from the fact that the squared magnitude operation $|\tilde{x}|^2 = \tilde{x}\tilde{x}^*$ is bandlimited, but the magnitude operation $|\tilde{x}| = \sqrt{\tilde{x}\tilde{x}^*}$ is not bandlimited. The non-bandlimited nature of even-order terms allows modeling of spectral regrowth over a wider frequency range with only a few terms, given that the sampling rate is high enough to avoid aliasing.

D. Representation by Odd-Order Terms

Since even-order passband basis functions of the form $x|x|^{p-1}$ with even p are odd-symmetric functions, they can be approximated by a series of conventional odd-order passband basis functions of the form x^p with odd p. This means that even-order terms contain contributions from all odd-order terms, which explains the non-bandlimited nature as well as the potential for improved modeling accuracy.

VI. CONCLUSION

The widely-used practice of excluding even-order terms from polynomial baseband models results from the implicit assumption of passband polynomial basis functions of the form x^p, which are odd-symmetric functions for odd p and even-symmetric functions for even p. If alternative passband basis functions of the form $x|x|^{p-1}$ are considered which are odd-symmetric functions for all p, the presence of even-order terms in the baseband model can be explained.

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