Baseband Volterra Filters with Even-Order Terms: Theoretical Foundation and Practical Implications

Harald Enzinger*, Karl Freiberger*, Gernot Kubin* and Christian Vogel*†
*Signal Processing and Speech Communication Laboratory, Graz University of Technology, Austria
†FH Joanneum - University of Applied Sciences, Austria
Email: enzinger@tugraz.at, freiberger@tugraz.at, g.kubin@ieee.org, c.vogel@ieee.org

***Abstract*—The baseband Volterra series is a general approach to model nonlinear passband systems like radio frequency power amplifiers in equivalent baseband. In the present paper, we review the derivation of the baseband Volterra series using a compact vector notation and show that it only includes odd-order terms. After that, we present a new derivation which shows that by assuming modified basis functionals in the passband, one obtains a baseband Volterra series which also includes even-order terms. By simulations, we demonstrate that the inclusion of the proposed even-order basis functionals improves the performance of behavioral modeling and digital predistortion and decreases the condition number of the regression matrix.**

I. Introduction

The Volterra series is a general approach to approximate the input-output behavior of nonlinear systems with memory [1]. For behavioral modeling of radio frequency power amplifiers the Volterra series is commonly used in its baseband form which models the distortion near the carrier frequency, but excludes the distortion at higher harmonics. Benedetto et al. [2] were the first who derived the baseband form of the Volterra series, showing that its set of basis functionals only includes odd-order terms. Due to the generality of the Volterra series this lead many researchers to exclude even-order terms from polynomial baseband models [3]. Experimentally, however, it was shown that even-order terms can improve the accuracy of polynomial baseband models [4]. Also generalized versions of the baseband Volterra series with even-order terms were proposed [5]–[7], justified mainly by physical intuition.

In contrast to [5]–[7], we present a purely mathematical analysis of even-order terms in baseband Volterra models. As in [8], where we analyzed memoryless and quasi-memoryless models, we derive explicit passband-baseband transformation pairs which form a theoretical basis for even-order terms in baseband Volterra models. Our contributions are:

- A review of the derivation by Benedetto et al. [2], using concise vector notation which simplifies the presentation.
- A new derivation showing an explicit relation between a passband Volterra series with modified basis functionals and a baseband Volterra series with even-order terms.
- A numerical validation, demonstrating improved performance for behavioral modeling and digital predistortion.

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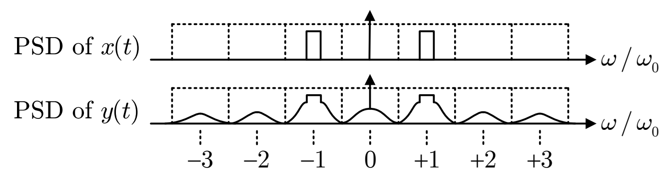


Fig. 1. Spectra of input and output signals of a nonlinear passband system.

II. Passband Volterra Series

The Volterra series is a nonlinear operator which maps a real input signal $x(t)$ to a real output signal $y(t)$ by a series of p-dimensional convolution integrals, given by

$$y(t) = \sum_{p=1}^{P} \int_{\mathbb{R}^p} h_p(\boldsymbol{\tau}_p) \prod_{i=1}^{p} x(t-\tau_i)\, \mathrm{d}\boldsymbol{\tau}_p \tag{1}$$

For notational simplicity, we use vector notation for the time-lags defined by $\boldsymbol{\tau}_p = [\tau_1, \tau_2, \ldots, \tau_p]^{\mathrm{T}}$. If the Volterra series is used to model a nonlinear passband system like a radio frequency power amplifier, the input signal is given by

$$x(t) = \frac{1}{2}\tilde{x}(t)e^{+j\omega_0 t} + \frac{1}{2}\tilde{x}^*(t)e^{-j\omega_0 t} \tag{2}$$

where $\tilde{x}(t)$ is the complex baseband signal, $\tilde{x}^*(t)$ is its complex conjugate and ω_0 is the angular carrier frequency. By inserting (2) into (1) and expanding terms, the output signal can be expressed as a series of modulated harmonics given by

$$y(t) = \sum_{p=1}^{P} \sum_{\substack{k=-p \\ \text{steps of 2}}}^{+p} \frac{1}{2}\tilde{y}_{p,k}(t)e^{jk\omega_0 t}. \tag{3}$$

The spectra of (2) and (3) are illustrated in Fig. 1. Typically, only the output near the carrier frequency is relevant. This leads to the baseband Volterra series that maps the complex input signal $\tilde{x}(t)$ to the complex output signal $\tilde{y}(t)$, which represents the modulation at the first harmonic. In section III we present the derivation of the baseband Volterra series showing that it only contains odd-order terms. After that, in section IV, we present a new derivation which shows that by assuming modified basis functionals in passband, one obtains a baseband Volterra series which includes even-order terms. In section V we present numerical simulations which validate the derived models, followed by a conclusion in section VI.

$$y(t) = \sum_{p=1}^{P} \sum_{\substack{k=-p \\ \text{steps of 2}}}^{+p} \int_{\mathbb{R}^p} \frac{1}{2^p} \binom{p}{\frac{p+k}{2}} h_p(\boldsymbol{\tau}_p) \prod_{i=1}^{\frac{p+k}{2}} \tilde{x}(t-\tau_i) e^{+j\omega_0(t-\tau_i)} \prod_{l=\frac{p+k}{2}+1}^{p} \tilde{x}^*(t-\tau_l) e^{-j\omega_0(t-\tau_l)} \, \mathrm{d}\boldsymbol{\tau}_p \tag{14}$$

III. Baseband Volterra Series

To simplify the derivation of the baseband Volterra series, we divide the real Volterra series from (1) into the sum

$$y(t) = \sum_{p=1}^{P} y_p(t) \tag{4}$$

with the p-th order convolution integrals given by

$$y_p(t) = \int_{\mathbb{R}^p} h_p(\boldsymbol{\tau}_p) \, \Phi_p(t, \boldsymbol{\tau}_p) \, \mathrm{d}\boldsymbol{\tau}_p \tag{5}$$

and the polynomial basis functionals given by

$$\Phi_p(t, \boldsymbol{\tau}_p) = \prod_{i=1}^{p} x(t-\tau_i) \, . \tag{6}$$

We define the analytic signal

$$\mathring{x}_{(t)}^{(c)} = \left[\frac{1}{2} \tilde{x}(t) e^{j\omega_0 t} \right]^{(*c)} \tag{7}$$

where c is a binary variable indicating not-conjugated terms ($c = 0$) and conjugated terms ($c = 1$). Using (7), the input signal from (2) can be represented by the two-element sum

$$x(t) = \sum_{c \in \mathbb{B}} \mathring{x}_{(t)}^{(c)} \tag{8}$$

with $\mathbb{B} = \{0, 1\}$. Inserting (8) into (6) we get

$$\Phi_p(t, \boldsymbol{\tau}_p) = \prod_{i=1}^{p} \sum_{c \in \mathbb{B}} \mathring{x}_{(t-\tau_i)}^{(c)} \, . \tag{9}$$

To express $y(t)$ as a sum of modulated harmonics, like in (3), it is necessary to expand (9) by moving the sum before the product. This transforms a p-term product of two-term sums like in (9) into a 2^p-term sum of p-term products, given by

$$\Phi_p(t, \boldsymbol{\tau}_p) = \sum_{\boldsymbol{c}_p \in \mathbb{B}^p} \prod_{i=1}^{p} \mathring{x}_{(t-\tau_i)}^{(c_i)} \, . \tag{10}$$

Inserting (10) into (5) and pulling out the sum, we get

$$y_p(t) = \sum_{\boldsymbol{c}_p \in \mathbb{B}^p} \int_{\mathbb{R}^p} h_p(\boldsymbol{\tau}_p) \prod_{i=1}^{p} \mathring{x}_{(t-\tau_i)}^{(c_i)} \, \mathrm{d}\boldsymbol{\tau}_p \, . \tag{11}$$

To further simplify (11), we note that the sum over the 2^p terms in $\mathbb{B}^p$ contains many vectors $\boldsymbol{c}_p$ which only differ by a permutation of elements. The result of the integral in (11) is, however, invariant to a permutation of elements in $\boldsymbol{c}_p$. This invariance is not based on a specific assumption on the structure of the Volterra kernels, it only requires that the integration covers the whole support of the kernels. Due to this invariance, we can reduce the number of terms from 2^p to $p+1$ by only summing over the elements $\boldsymbol{c}_p$ which consist of q zeros, followed by $p-q$ ones and multiplying each term by the number of equivalent vectors $\boldsymbol{c}_p$ leading to

$$y_p(t) = \sum_{q=0}^{p} \binom{p}{q} \int_{\mathbb{R}^p} h_p(\boldsymbol{\tau}_p) \prod_{i=1}^{p} \mathring{x}_{(t-\tau_i)}^{(i>q)} \, \mathrm{d}\boldsymbol{\tau}_p \, . \tag{12}$$

Next we change the summation index from q, which represents the number of not-conjugated terms, to k, which represents the difference of not-conjugated terms and conjugated terms, i.e., $k = q - (p-q) = 2q - p$ or $q = \frac{p+k}{2}$, resulting in

$$y_p(t) = \sum_{\substack{k=-p \\ \text{steps of 2}}}^{+p} \binom{p}{\frac{p+k}{2}} \int_{\mathbb{R}^p} h_p(\boldsymbol{\tau}_p) \prod_{i=1}^{p} \mathring{x}_{(t-\tau_i)}^{\left(i>\frac{p+k}{2}\right)} \, \mathrm{d}\boldsymbol{\tau}_p \, . \tag{13}$$

Now we substitute (7) into (13) and (13) into (4) which leads to (14) on top of this page. Since k represents the difference between not-conjugated and conjugated terms, we can pull out $e^{jk\omega_0 t}$ from (14) and write it as the harmonic series

$$y(t) = \sum_{p=1}^{P} \sum_{\substack{k=-p \\ \text{steps of 2}}}^{+p} \frac{1}{2} \tilde{y}_{p,k}(t) e^{jk\omega_0 t} \tag{15}$$

with the modulation of the harmonics given by

$$\tilde{y}_{p,k}(t) = \int_{\mathbb{R}^p} \tilde{h}_{p,k}(\boldsymbol{\tau}_p) \, \tilde{\Phi}_{p,k}(t, \boldsymbol{\tau}_p) \, \mathrm{d}\boldsymbol{\tau}_p \, . \tag{16}$$

The baseband kernels are scaled passband kernels given by

$$\tilde{h}_{p,k}(\boldsymbol{\tau}_p) = h_p(\boldsymbol{\tau}_p) \, \lambda_{p,k} \, e^{j\varphi_{p,k}(\boldsymbol{\tau}_p)} \tag{17}$$

with the magnitude of the scaling given by

$$\lambda_{p,k} = \frac{1}{2^{p-1}} \binom{p}{\frac{p+k}{2}} \tag{18}$$

and the argument of the scaling given by

$$\varphi_{p,k}(\boldsymbol{\tau}_p) = \omega_0 \left(-\sum_{i=1}^{\frac{p+k}{2}} \tau_i + \sum_{l=\frac{p+k}{2}+1}^{p} \tau_l \right) \, . \tag{19}$$

The baseband basis functionals are given by

$$\tilde{\Phi}_{p,k}(t, \boldsymbol{\tau}_p) = \prod_{i=1}^{\frac{p+k}{2}} \tilde{x}(t-\tau_i) \prod_{l=\frac{p+k}{2}+1}^{p} \tilde{x}^*(t-\tau_l) \, . \tag{20}$$

According to the summations in (15), p and k must have the same parity. Therefore the baseband Volterra series which models the modulation at $k = 1$ only contains odd-order terms.

IV. Baseband Volterra Series with Even-Order Terms

A. Motivation for Even-Order Terms in Baseband

In the previous section we have shown that the first harmonic output of a passband Volterra series can be described by a baseband Volterra series with only odd-order terms. For behavioral modeling and digital predistortion, however, one seeks a baseband model which is able to efficiently approximate an unknown passband system. For this task, the restriction to odd-orders might not give the best results. Therefore we are interested in deriving a baseband Volterra series which includes even-order terms. As a starting point, we use a result from [8] where we have shown that for memoryless systems, the set of odd-order basis functions given by

$$\tilde{\Phi}_1(t) = \tilde{x}(t)\,, \tag{21}$$

$$\tilde{\Phi}_3(t) = \tilde{x}(t)\,\tilde{x}(t)\,\tilde{x}^*(t) = \tilde{x}(t)\,|\tilde{x}(t)|^2\,, \tag{22}$$

$$\tilde{\Phi}_5(t) = \tilde{x}(t)\,\tilde{x}(t)\,\tilde{x}(t)\,\tilde{x}^*(t)\,\tilde{x}^*(t) = \tilde{x}(t)\,|\tilde{x}(t)|^4\,, \tag{23}$$

can be generalized to the passband-baseband transformation

$$\Phi_p(t) = x(t)\,|x(t)|^{p-1} \;\leftrightarrow\; \tilde{\Phi}_p(t) = \lambda_{p,1}\,\tilde{x}(t)\,|\tilde{x}(t)|^{p-1} \tag{24}$$

which is valid for both odd and even values of p. For odd p, the passband basis functions are monomials x^p, but for even p, they are modified monomials $\operatorname{sign}(x)\,x^p$. The modified monomials are even-order terms, but odd-symmetric functions and therefore produce output at odd harmonics. Similar to the memoryless case, we will investigate a passband Volterra series with modified even-order basis functionals given by

$$\Phi_p(t,\boldsymbol{\tau}_p) = \prod_{i=1}^{p-1} x(t-\tau_i)\,|x(t-\tau_p)|\,. \tag{25}$$

and derive the corresponding basis functionals in baseband.

B. Derivation of Even-Order Terms in Baseband

In the following we will assume that p is even and for compact notation, we define $q = p-1$ which is therefore odd. By inserting (25) into (5) and dividing the p-dimensional integral into a q-dimensional integral over $\boldsymbol{\tau}_q$ and a one-dimensional integral over τ_p, the p-th order output is

$$y_p(t) = \int_{\mathbb{R}}\int_{\mathbb{R}^q} h_p(\boldsymbol{\tau}_q,\tau_p)\,\Phi_q(t,\boldsymbol{\tau}_q)\,|x(t-\tau_p)|\,\mathrm{d}\boldsymbol{\tau}_q\mathrm{d}\tau_p\,. \tag{26}$$

To simplify (26), we separate the expression inside the one-dimensional integral into two factors such that we can write

$$y_p(t) = \int_{\mathbb{R}} f_q(t,\tau_p)\,g(t,\tau_p)\,\mathrm{d}\tau_p \tag{27}$$

with the first factor given by

$$f_q(t,\tau_p) = \int_{\mathbb{R}^q} h_p(\boldsymbol{\tau}_q,\tau_p)\,\Phi_q(t,\boldsymbol{\tau}_q)\,\mathrm{d}\boldsymbol{\tau}_q \tag{28}$$

and the second factor given by

$$g(t,\tau_p) = |x(t-\tau_p)|\,. \tag{29}$$

Since (28) is a conventional odd-order Volterra integral, we can reuse the harmonic series from section III given by

$$f_q(t,\tau_p) = \sum_{\substack{k_1=-q\\ \text{steps of 2}}}^{+q} \frac{1}{2}\tilde{f}_{q,k_1}(t,\tau_p)e^{jk_1\omega_0 t} \tag{30}$$

with the modulation of the harmonics given by

$$\tilde{f}_{q,k}(t,\tau_p) = \int_{\mathbb{R}^q} \tilde{h}_{p,k}(\boldsymbol{\tau}_q,\tau_p)\,\tilde{\Phi}_{q,k}(t,\boldsymbol{\tau}_q)\,\mathrm{d}\boldsymbol{\tau}_q\,. \tag{31}$$

The term in (29) is an even-symmetric memoryless passband nonlinearity and can be represented by the harmonic series [9]

$$g(t,\tau_p) = \sum_{\substack{k_2=-\infty\\ k_2 \text{ is even}}}^{+\infty} \tilde{g}_{k_2}(t,\tau_p)e^{jk_2\omega_0 t} \tag{32}$$

with the modulation of the harmonics given by

$$\tilde{g}_k(t,\tau_p) = \frac{4}{\pi}\frac{(-1)^{\frac{k}{2}+1}}{k^2-1}\,|\tilde{x}(t-\tau_p)|\,e^{jk(\angle\tilde{x}(t-\tau_p)-\omega_0\tau_p)}\,. \tag{33}$$

By inserting (30) and (32) into (27) we get the harmonic series

$$y_p(t) = \sum_{\substack{k_1=-q\\ \text{steps of 2}}}^{+q}\;\sum_{\substack{k_2=-\infty\\ k_2 \text{ is even}}}^{+\infty} \frac{1}{2}\tilde{y}_{p,k_1,k_2}(t,\tau_p)e^{j(k_1+k_2)\omega_0 t} \tag{34}$$

with the modulation of the harmonics given by

$$\tilde{y}_{p,k_1,k_2}(t) = \int_{\mathbb{R}} \tilde{f}_{q,k_1}(t,\tau_p)\,\tilde{g}_{k_2}(t,\tau_p)\,\mathrm{d}\tau_p\,. \tag{35}$$

To transform (34) into a harmonic series with a single index k, we apply the substitution $k_2 = k - k_1$, resulting in

$$y_p(t) = \sum_{\substack{k=-\infty\\ k \text{ is odd}}}^{+\infty} \frac{1}{2}\tilde{y}_{p,k}(t,\tau_p)e^{jk\omega_0 t} \tag{36}$$

with the modulation of the harmonics given by

$$\tilde{y}_{p,k}(t) = \sum_{\substack{k_1=-q\\ \text{steps of 2}}}^{+q} \int_{\mathbb{R}} \tilde{f}_{q,k_1}(t,\tau_p)\,\tilde{g}_{k-k_1}(t,\tau_p)\,\mathrm{d}\tau_p\,. \tag{37}$$

By inserting (31) and (33) into (37) we can represent it as

$$\tilde{y}_{p,k}(t) = \sum_{\substack{k_1=-q\\ \text{steps of 2}}}^{+q} \int_{\mathbb{R}^p} \tilde{h}^{(k_1)}_{p,k}(\boldsymbol{\tau}_p)\tilde{\Phi}^{(k_1)}_{p,k}(t,\boldsymbol{\tau}_p)\,\mathrm{d}\boldsymbol{\tau}_p \tag{38}$$

with the baseband kernels given by

$$\tilde{h}^{(k_1)}_{p,k}(\boldsymbol{\tau}_p) = h_p(\boldsymbol{\tau}_p)\,\lambda^{(k_1)}_{p,k}\,e^{j\varphi^{(k_1)}_{p,k}(\boldsymbol{\tau}_p)}\,, \tag{39}$$

the magnitude and argument of the scaling given by

$$\lambda^{(k_1)}_{p,k} = \lambda_{q,k_1}\,\frac{4}{\pi}\frac{(-1)^{\frac{k-k_1}{2}+1}}{(k-k_1)^2-1}\,, \tag{40}$$

$$\varphi^{(k_1)}_{p,k}(\boldsymbol{\tau}_p) = \varphi_{q,k_1}(\boldsymbol{\tau}_q) - (k-k_1)\omega_0\tau_p \tag{41}$$

and the baseband basis functionals given by

$$\tilde{\Phi}^{(k_1)}_{p,k}(t,\boldsymbol{\tau}_p) = \tilde{\Phi}_{q,k_1}(t,\boldsymbol{\tau}_q)\,|\tilde{x}(t-\tau_p)|\,e^{j(k-k_1)\angle\tilde{x}(t-\tau_p)}\,. \tag{42}$$

TABLE I

SUMMARY OF DISCRETE-TIME BASEBAND VOLTERRA FILTER WITH EVEN-ORDER TERMS.

(a) Computation of basis functionals.
$\tilde{\Phi}_1[n,\boldsymbol{m}_1] = \tilde{x}_{n-m_1}$
$\tilde{\Phi}_2[n,\boldsymbol{m}_2] = \tilde{x}_{n-m_1}\left|\tilde{x}_{n-m_2}\right|$
$\tilde{\Phi}_3[n,\boldsymbol{m}_3] = \tilde{x}_{n-m_1}\ \tilde{x}_{n-m_2}\ \ \tilde{x}^*_{n-m_3}$
$\tilde{\Phi}_4[n,\boldsymbol{m}_4] = \tilde{x}_{n-m_1}\ \tilde{x}_{n-m_2}\ \ \tilde{x}^*_{n-m_3}\ \left|\tilde{x}_{n-m_4}\right|$
$\tilde{\Phi}_5[n,\boldsymbol{m}_5] = \tilde{x}_{n-m_1}\ \tilde{x}_{n-m_2}\ \ \tilde{x}_{n-m_3}\ \ \tilde{x}^*_{n-m_4}\ \ \tilde{x}^*_{n-m_5}$
$\tilde{\Phi}_6[n,\boldsymbol{m}_6] = \tilde{x}_{n-m_1}\ \tilde{x}_{n-m_2}\ \ \tilde{x}_{n-m_3}\ \ \tilde{x}^*_{n-m_4}\ \ \tilde{x}^*_{n-m_5}\ \left|\tilde{x}_{n-m_6}\right|$

(b) Computation of basis matrix.
$\tilde{\boldsymbol{\Phi}}_p^{(i)} = \left[\tilde{\Phi}_p\left[0,\boldsymbol{m}_p^{(i)}\right],\tilde{\Phi}_p\left[1,\boldsymbol{m}_p^{(i)}\right],\ldots,\tilde{\Phi}_p\left[N-1,\boldsymbol{m}_p^{(i)}\right]\right]^{\mathrm{T}}$
$\tilde{\boldsymbol{\Phi}}_p \ \ = \left[\tilde{\boldsymbol{\Phi}}_p^{(1)},\tilde{\boldsymbol{\Phi}}_p^{(2)},\ldots,\tilde{\boldsymbol{\Phi}}_p^{(C_p)}\right]$
$\tilde{\boldsymbol{\Phi}} \ \ \ = \left[\tilde{\boldsymbol{\Phi}}_1,\tilde{\boldsymbol{\Phi}}_2,\ldots,\tilde{\boldsymbol{\Phi}}_P\right]$

C. Selection of Even-Order Terms in Baseband

The derivation in the previous subsection verifies that the modified passband basis functionals in (25) produce output at odd harmonics, as shown by the harmonic series in (36). However, the modulation of the first harmonic is not described by a single baseband Volterra integral, but by the sum over p baseband Volterra integrals given in (38). Correspondingly there are also p baseband basis functionals given in (42), where the index k_1 is an odd number between $p-1$ and $p+1$.

Since our goal is not an exact reproduction of the assumed passband basis functional, we do not require to use all of the derived baseband basis functionals in (42). To keep the complexity low, we only choose one of them, given by $k_1 = k$. This eliminates the exponential term in (42) and results in our definition of even p-th order baseband basis functionals by

$$\tilde{\Phi}_{p,k}(t,\boldsymbol{\tau}_p) = \tilde{\Phi}_{p-1,k}(t,\boldsymbol{\tau}_{p-1})\left|\tilde{x}(t-\tau_p)\right|, \qquad (43)$$

with the odd-order basis functionals $\tilde{\Phi}_{p-1,k}$ given in (20).

D. Implementation of Baseband Volterra Filters

To transform the continuous-time Volterra series into a causal discrete-time Volterra filter, we replace the integration over the time-lag vectors $\boldsymbol{\tau}_p \in \mathbb{R}^p$ by a finite summation over the sample-lag vectors $\boldsymbol{m}_p \in \mathcal{M}_p$ which results in

$$\tilde{y}[n] = \sum_{p\in\mathcal{P}}\sum_{\boldsymbol{m}_p\in\mathcal{M}_p}\tilde{h}_p[\boldsymbol{m}_p]\,\tilde{\Phi}_p[n,\boldsymbol{m}_p], \qquad (44)$$

with the odd-order basis, obtained from (20), given by

$$\tilde{\Phi}_p[n,\boldsymbol{m}_p] = \prod_{i=1}^{\frac{p+1}{2}}\tilde{x}[n-m_i]\prod_{l=\frac{p+3}{2}}^{p}\tilde{x}^*[n-m_l], \qquad (45)$$

and the even-order basis, obtained from (43), given by

$$\tilde{\Phi}_p[n,\boldsymbol{m}_p] = \tilde{\Phi}_{p-1}[n,\boldsymbol{m}_{p-1}]\left|\tilde{x}[n-m_p]\right|. \qquad (46)$$

An overview of the basis functionals is given in Table I (a), where we use the sample-index notation $\tilde{x}_n = \tilde{x}[n]$.

The kernel structure of the discrete-time Volterra filter is defined by the set of orders $\mathcal{P}$ and the corresponding sets of sample-lag vectors $\mathcal{M}_p$. For a given p-th order memory depth $M_p \in \mathbb{N}_0$, we define the set of sample-lag vectors $\mathcal{M}_p$ as a subset of the p-dimensional hypercube $\{\boldsymbol{m}_p \in \mathbb{N}_0^p | m_i \leq M_p\}$. We only use a subset of this hypercube, since permutations of sample-lags m_i within not-conjugated terms and permutations of sample-lags m_l within conjugated terms, result in the same output of (45). To generate $\mathcal{M}_p$ without the redundant terms, we note that in $\tilde{\Phi}_p$, the number of not-conjugated terms is

$$K_p^{(1)} = \left\lfloor\frac{p+1}{2}\right\rfloor, \qquad (47)$$

the number of conjugated terms is

$$K_p^{(2)} = \left\lfloor\frac{p-1}{2}\right\rfloor, \qquad (48)$$

and the number of magnitude terms is

$$K_p^{(3)} = 2\,\mathrm{rem}\,(p+1,2)\;, \qquad (49)$$

where $\mathrm{rem}(p{+}1,2)$ returns the remainder of $\frac{p+1}{2}$. Defining the set of first-order sample lag vectors by $\mathcal{M}_1 = \{0,1,\ldots,M_p\}$, the higher orders are generated by the Cartesian product

$$\mathcal{M}_p = \mathcal{M}_p^{(1)} \times \mathcal{M}_p^{(2)} \times \mathcal{M}_p^{(3)} \qquad (50)$$

where $\mathcal{M}_p^{(1)}$ is the set of $K_p^{(1)}$-multicombinations of $\mathcal{M}_1$, $\mathcal{M}_p^{(2)}$ is the set of $K_p^{(2)}$-multicombinations of $\mathcal{M}_1$ and $\mathcal{M}_p^{(3)}$ is the empty set for odd p and $\mathcal{M}_1$ for even p. The cardinality

$$C_p = |\mathcal{M}_p| = \left(\!\!\binom{M_p+1}{K_p^{(1)}}\!\!\right)\left(\!\!\binom{M_p+1}{K_p^{(2)}}\!\!\right)\left(M_p+1\right)^{K_p^{(3)}} \qquad (51)$$

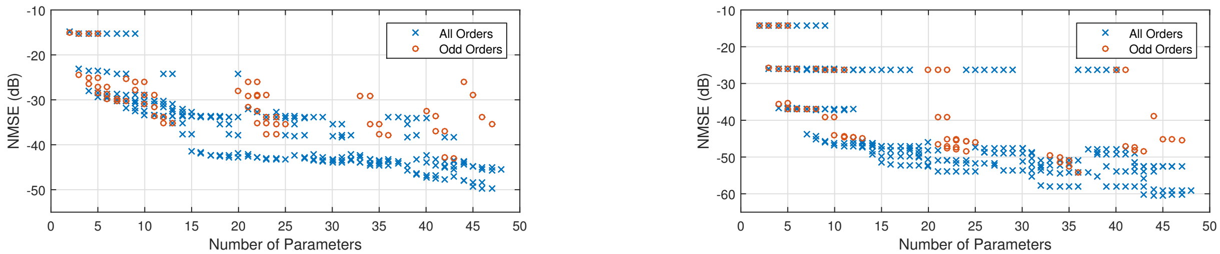
with $\left(\!\binom{M}{K}\!\right) = \binom{M+K-1}{K}$ represents the number of p-th order basis vectors, or equivalently, the number of parameters in the discrete-time p-th order kernel with memory depth M_p. The total number of parameters is $C = \sum_{p\in\mathcal{P}} C_p$. The mapping

$$\tilde{\boldsymbol{x}} = [\tilde{x}_0,\tilde{x}_1,\ldots,\tilde{x}_{N-1}]^{\mathrm{T}} \mapsto \tilde{\boldsymbol{y}} = [\tilde{y}_0,\tilde{y}_1,\ldots,\tilde{y}_{N-1}]^{\mathrm{T}} \qquad (52)$$

is implemented by the matrix-vector product $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{\Phi}}\,\tilde{\boldsymbol{h}}$ with the $N \times C$ basis matrix $\tilde{\boldsymbol{\Phi}}$ computed according to Table I (b) where $\boldsymbol{m}_p^{(i)}$ is the i-th element in $\mathcal{M}_p$. The $C \times 1$ parameter vector $\tilde{\boldsymbol{h}}$ containing the vectorized kernel $\tilde{h}_p[\boldsymbol{m}_p]$ may be identified from a pair of input-output vectors $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ by

$$\tilde{\boldsymbol{h}} = \left(\tilde{\boldsymbol{\Phi}}^{\mathrm{H}}\tilde{\boldsymbol{\Phi}}\right)^{-1}\tilde{\boldsymbol{\Phi}}^{\mathrm{H}}\,\tilde{\boldsymbol{y}}\,. \qquad (53)$$

The condition number of $\tilde{\boldsymbol{\Phi}}^{\mathrm{H}}\tilde{\boldsymbol{\Phi}}$, which is the ratio of the largest to the smallest eigenvalue, quantifies the error-sensitivity.



(a) Behavioral modeling.

(b) Digital predistortion.

Fig. 2. Simulation results: We evaluated 277 configurations of baseband Volterra filters with orders up to 9, memory depths up to 3 and less than 50 parameters. The blue crosses represent configurations which include both odd and even-order terms, the red circles represent configurations which only include odd orders.

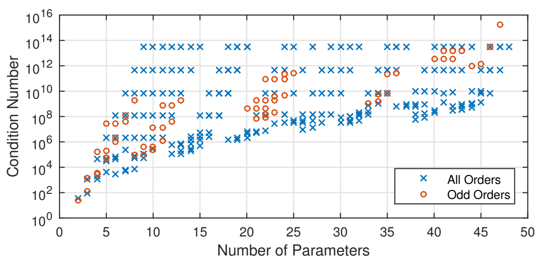


Fig. 3. Condition number of $\tilde{\boldsymbol{\Phi}}^{\mathrm{H}}\tilde{\boldsymbol{\Phi}}$, constructed from $\tilde{\boldsymbol{x}}$ which was used for the identification of the baseband Volterra filter during behavioral modeling.

V. Simulations

To validate the proposed even-order basis functionals, we implemented the baseband Volterra filter with even-order terms and used it for behavioral modeling and digital predistortion of a quasi-memoryless nonlinearity between two linear filters. Both linear filters have a single pole at 0.2 and a gain of 0.8. The nonlinearity is an extended Rapp model [10] with

$$\begin{aligned} a_{\mathrm{g}} &= 10\,, \quad & b_{\mathrm{g}} &= 10\,, \quad & c_{\mathrm{g}} &= 2\,, \\ a_{\varphi} &= \frac{\pi}{10}\,, \quad & b_{\varphi} &= -\frac{\pi}{5}\,, \quad & c_{\varphi} &= 0\,, \quad & d_{\varphi} = +\frac{\pi}{5}\,. \end{aligned}$$

For identification and evaluation we used dense random-phase multitone signals with an oversampling ratio of 5, a crest factor of 10 dB and lengths of 10 and 100 kSamples, respectively. The output backoff for behavioral modeling was 6 dB, whereas for digital predistortion it was 12 dB. The architecture for digital predistortion was indirect learning as described in [11].

We simulated two sets of baseband Volterra filters with less than 50 parameters. For the first set we used all orders from 1 up to $P \leq 9$ and all multicombinations of memory depths with $M_{p+1} \leq M_p \leq 3$ resulting in 216 configurations. For the second set, we used only odd orders with the same constraints resulting in 61 configurations. The performance in terms of normalized mean square error is shown in Fig. 2, which shows that by including even-order terms, an improvement by several dB can be achieved at the same number of parameters.

The improved performance of baseband Volterra filters with even-order terms may be explained by a reduced condition number of the regression matrix as shown in Fig. 3.

VI. Conclusion

We presented a compact derivation of the baseband Volterra series and showed that by assuming modified basis functionals in passband, we can derive even-order terms in baseband. In future work, the fast time-domain computation described in [12] may also be applied to the baseband Volterra series.

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