

Static Nonlinearities

Universal Approximators and System Identification

Problem 2.1

Generate a sequence $x[n]$ of 500 samples, spaced uniformly in the interval $[-1, 1]$. You can use `linspace` to generate this sequence. Using any one of the nonlinearities `n11.m`, `n12.m`, and `n13.m`, generate the corresponding output sequence $y[n]$. Plotting $y[n]$ as a function of $x[n]$ allows to visualize the nonlinear mapping involved. Use `rbf.m` to identify (approximate) this nonlinearity using Radial Basis Functions (RBF). The quality of the approximation can be assessed using

$$Q_2 = 10 \log_{10} \frac{\sum_n |y_{orig}[n] - y_{reconst}[n]|^2}{\sum_n |y_{orig}[n]|^2}$$

(a) Select 4 centers uniformly spaced between -1 and 1 . Set the width of RBFs to 0.1 . Compute the corresponding Q_2 -value. Try to vary the widths and centers of the RBFs. How does this affect the performance? Find parameter values that ensure the best performance with respect to Q_2 using 4 RBF functions.

(b) Once the optimal parameters are found, increase the number of centers to 9, 15, 25, 50, and 100. Compute the corresponding Q_2 -values. How does the number of centers affect the performance? What about the kernel conditioning if

1. the number of the centers grows and the width of the RBFs is kept constant?
2. the width of the kernels grows, while the number of RBFs is fixed?

Problem 2.2

The function `n14.m` implements a quantizer with quantization steps $\Delta = 0.5$. Generate an input signal $x[n]$ limited to the interval $[-1, 1]$ and perform the following tasks:

(a) Using the provided `rbf.m` function, find the parameters (i.e., centers and widths of the RBFs) that ensure an approximation quality of $Q_2 \leq -20dB$.

(b) Perform the same approximation as in (a) using `sigmoid.m` and `fourier.m`. How many centers do you need in this case to achieve the same performance?

(c) What is the best kernel function for the given nonlinearity, what is the worst? Find an RBF that would be suitable for this task. Explain your answers.

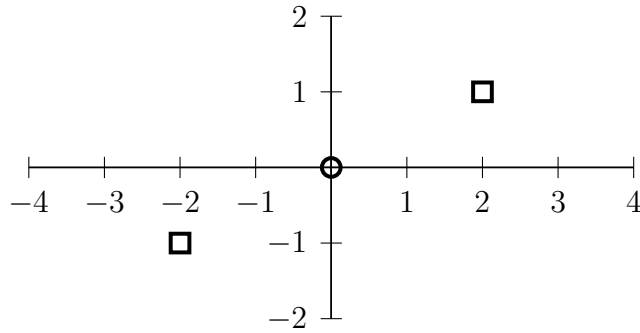


Figure 1: 'Squares' vs. 'Circles'

Problem 2.3

Consider two data sets – circles and squares, shown in Fig.1.

(a) Using RBFs, find the *analytical* expression for the function $f(\cdot)$ that classifies the points belonging to the same class as follows:

$$\begin{aligned} f(\text{'squares'}) &= a \\ f(\text{'circles'}) &= -a \end{aligned} \quad (1)$$

or, more formally,

$$f(\mathbf{x}) = \begin{cases} a & \text{if } \mathbf{x} \in \{(-2, -1), (2, 1)\} \\ -a & \text{if } \mathbf{x} = (0, 0) \end{cases} \quad (2)$$

where a is an arbitrary nonzero constant. Explain your choice of the RBF parameters.

(b) Using Matlab's `randn` function, generate normally distributed random samples for each of the two classes, centered around $(0, 0)$ for 'circles', and around $(-2, -1)$ and $(2, 1)$ for 'squares', respectively. Keep the variance of the generated samples low enough to avoid overlapping. Use the generated samples to check if the function $f(\cdot)$ can correctly predict (classify) the class of the data points.

Problem 2.4

The provided file `amplifier.mat` contains input/output values ($x[n]$ and $y[n]$, resp.) for an amplifier system. Prior knowledge suggests that the amplifier is a static nonlinearity. Note that there is additive noise $\eta[n]$ (zero-mean); thus perfect approximation is not desired.

(a) Have a look at the nonlinearity

$$y[n] \approx A(x[n]) \quad (3)$$

i.e., plot $y[n]$ as a function of $x[n]$. Also calculate the Q_2 -value of $y[n]$ based on $x[n]$ ($x[n]$ is the "original" signal).

(b) Fit a polynomial in the least-squares sense to the function $A(x[n])$. Remember that you do not want to model the noise, but only the hidden amplifier behaviour. Choose a suitable order for your polynomial! **Hint:** Have a look at the functions `polyfit` and `polyval`!

(c) Now model the *inverse* system \hat{A}^{-1} using a polynomial approximation and implement the post-distorter system shown in Fig. 2. If the linearization works, what do you expect at the output $z[n]$?

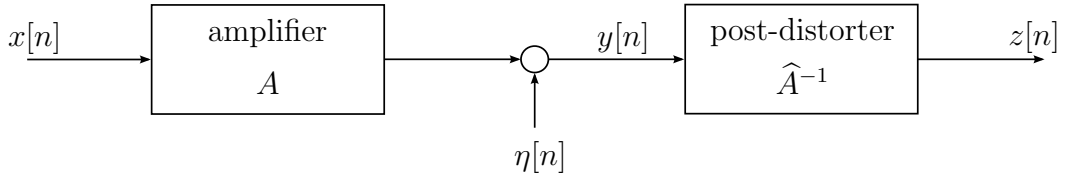


Figure 2: Amplifier with post-distorter.

(d) Create input/output plots for polynomial orders 0 through 4 and calculate the corresponding Q_2 -values. Plot $y[n]$ and $z[n]$ over $x[n]$ and also add a 45° line for each polynomial order. Based on the Q_2 -values, which order would you choose?

Harmonic Analysis of Static Nonlinearities

Problem 2.5

For the given input and output of a nonlinear system:

$$x[n] = \cos(\theta n) = 0.5(e^{j\theta n} + e^{-j\theta n})$$

$$y[n] = f(x[n])$$

- (a) compute the harmonic distortions (frequencies) when $f(x[n]) = 1 - e^{-x[n]}$.
- (b) Plot the nonlinearity $f(x[n])$ for the range of input values $x[n] \in [-2.5, 2.5]$ overlaid with its third order Taylor expansion around zero.

Problem 2.6

- (a) Compute the harmonic distortions for the following nonlinear system

$$y[n] = 1 - x[n] + 3x[n]^2 - x[n]^4$$

when the input signal is $x[n] = \cos(\theta n)$. Support the obtained analytical results with the corresponding Matlab simulations. **Hint:** Use Chebyshev polynomials to obtain the solution.

Problem 2.7

Let's reconsider the amplifier from Problem 2.4. Assuming that higher-order terms can be neglected, we model the static nonlinear behavior of the amplifier as a third-order polynomial

$$y(x[n]) \approx a_1x[n] + a_2x^2[n] + a_3x^3[n] \quad (4)$$

Let the input signal be $x[n] = A \cos(\theta n)$. The nonlinearity of the amplifier leads to harmonic distortion at the output. A popular figure of merit for characterizing the linearity of amplifiers is the *Third-order Intercept point* IP_3 . For its definition, we look at the amplitude of the *fundamental frequency* component at the output. It contains first-order as well as third-order terms, i.e. A and A^3 . The IP_3 is defined as the input amplitude A for which these first- and third-order contributions to the fundamental frequency are equal.

(a) By plugging $x[n]$ in (4), derive a general formula for IP_3 in terms of the polynomial coefficients a_i .

(b) Now calculate the IP_3 for our amplifier. Assume that it is represented as the continuous function $y(x) = a \sinh(x)$. Therefore, calculate a third-order Taylor approximation of the amplifier to find the a_i , using

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f(c)}{dx^k} (x-c)^k = \sum_{k=0}^{\infty} a_k (x-c)^k \approx \sum_{k=0}^K a_k (x-c)^k.$$

around the *expansion center* $c = 0.75$. Support your result by a Matlab simulation, i.e., plot the linear and third-order system response (only for the fundamental frequency!) and look at the intersection point. What is the meaning of IP_3 ? Also describe the influence of the choice of the expansion center c on the result! **Hint:** The identity $a \sinh(x) = \ln(x + \sqrt{x^2 + 1})$ might be useful.

Statistical Analysis of Static Nonlinearities

Problem 2.8

Consider an exponentially distributed random variable X

$$f_X(x) = \lambda \exp(-\lambda x), \quad x \in X$$

- (a) Find the corresponding characteristic function $\Phi(ju)$.
- (b) Obtain the expression for moments and cumulants of X .

Problem 2.9

- (a) For $X \sim \mathcal{N}(0, \sigma^2)$, and $Y = aX^2 + b$, find mean μ_y , variance σ_y , and PDF $f_Y(y)$.
- (b) Let $X_1 \sim \mathcal{N}(0, \sigma^2)$ and $X_2 \sim \mathcal{N}(0, \sigma^2)$ be two independent random variables (i.i.d.), and let $Y = X_1^2 + X_2^2$. Find the PDF $f_Y(y)$ of the random variable Y .

Problem 2.10

Consider two zero-mean, circular symmetric, complex gaussian stochastic processes $x[n] = a[n] + jb[n]$ and $y[n] = c[n] + jd[n]$, where $a[n]$, $b[n]$, $c[n]$, and $d[n]$ are jointly independent and $\sim \mathcal{N}(0, \sigma^2)$. Find the cumulants of the power level

(a) $z[n] = |x[n]|^2 + |y[n]|^2$.

(b) $z[n] = \sum_{l=1}^L |x[l]|^2$. All samples of $x[n]$ are i.i.d. $\sim \mathcal{N}(0, \sigma^2)$.

Problem 2.11

(a) A random variable X is known to be zero mean, with central moments $\mu_r(X)$ and cumulants $c_r(X)$. Show that $c_3(X) = \mu_3(X)$ and $c_4(X) = \mu_4(X) - 3\mu_2^2(X)$

(b) Show that the odd-order moments of a zero-mean random variable X with a symmetric PDF, i.e. $f_X(x) = f_X(-x)$, are zero.